Math 2233 - Lecture 17 : Review Session for Exam 2

Agenda:

1. Two Basic types of 2nd Order Linear ODEs
2. Solving 2nd Order Linear ODEs
   - Homogeneous Case
   - Nonhomogeneous Case
3. Two Simple Cases
   - Constant Coefficient ODEs : $ay'' + by' + cy = 0$
   - Euler-type ODEs : $ax^2y'' + bxy' + cy = 0$
4. The Laplace Transform Method

Solving 2nd Order Linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{0}$$

$$y'' + p(x)y' + q(x)y = g(x) \neq 0 \tag{1}$$

Solving Homogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{0}$$

Solving Homogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{0}$$

- The Superposition Principle: If $y_1(x)$ and $y_2(x)$ are solutions of (0), then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (0)

Solving Homogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \qquad (0)$$

- The Superposition Principle: If $y_1(x)$ and $y_2(x)$ are solutions of (0), then $c_1 y_1(x) + c_2 y_2(x)$ is also a solution of (0)
- Reduction of Order: If $y_1(x)$ is a a solution of (0), then

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\,dx\right] dx$$

is a second independent solution of (0).

Solving Homogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{0}$$

- The Superposition Principle: If $y_1(x)$ and $y_2(x)$ are solutions of (0), then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (0)
- Reduction of Order: If $y_1(x)$ is a a solution of (0), then

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\, dx\right] dx$$

  is a second independent solution of (0).
- The general solution of (0) will be of the form

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

  where y_1, y_2 are two independent solutions.

Solving Homogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{0}$$

- The Superposition Principle: If $y_1(x)$ and $y_2(x)$ are solutions of (0), then $c_1 y_1(x) + c_2 y_2(x)$ is also a solution of (0)
- Reduction of Order: If $y_1(x)$ is a a solution of (0), then

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\,dx\right] dx$$

  is a second independent solution of (0).
- The general solution of (0) will be of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

  where y_1, y_2 are two independent solutions.
- "Independent" means

$$\begin{aligned} 0 \quad &\neq \quad W[y_1, y_2](x) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &\Leftrightarrow y_2(x) \neq \lambda y_1(x) \end{aligned}$$

Solving Nonhomogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

Solving Nonhomogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

- The form of the general solution will be

  $$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2 x$$

  where $y_p(x)$ is any particular solution of (1) and $y_1(x), y_2(x)$ are two independents solutions of the corresponding homogeneous ODE (0).

Solving Nonhomogeneous Second Order Linear ODEs

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

- The form of the general solution will be

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2 x$$

where $y_p(x)$ is any particular solution of (1) and $y_1(x), y_2(x)$ are two independents solutions of the corresponding homogeneous ODE (0).

- Variation of Parameters: If y_1, y_2 are two independent solutions of (0), then a particular solution $y_p(x)$ of (1) can be calculated as

$$y_p(x) = -y_1(x) \int \frac{y_2(x)\, g(x)}{W[y_1, y_2](x)} dx + y_2(x) \int \frac{y_1(x)\, g(x)}{W[y_1, y_2](x)} dx \tag{4}$$

Problem 1 of the Practice Exam

1. Explain in words and formulas how you would construct the general solution of $y'' + p(x)\,y' + q(x)\,y = g(x)$, given that $y_1(x)$ is a solution of $y'' + p(x)\,y' + q(x)\,y = 0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

Problem 1 of the Practice Exam

1. Explain in words and formulas how you would construct the general solution of $y'' + p(x)\,y' + q(x)\,y = g(x)$, given that $y_1(x)$ is a solution of $y'' + p(x)\,y' + q(x)\,y = 0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

- Step 1: Use Reduction of Order to find a second, independent, solution of the homogenous equation:

$$y_2 = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left(- \int^x p(s)\, ds \right) dx$$

Problem 1 of the Practice Exam

1. Explain in words and formulas how you would construct the general solution of $y'' + p(x)\,y' + q(x)\,y = g(x)$, given that $y_1(x)$ is a solution of $y'' + p(x)\,y' + q(x)\,y = 0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

- Step 1: Use Reduction of Order to find a second, independent, solution of the homogenous equation:

$$y_2 = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left(- \int^x p(s)\,ds \right) dx$$

- Step 2: Use Variations of Parameters to find a particular solution $y_p(x)$ of the inhomogeneous equation

$$y_p(x) = -y_1(x) \int \frac{y_2(x)\,g(x)}{W[y_1, y_2]} dx \quad + \quad y_2(x) \int \frac{y_1(x)\,g(x)}{W[y_1, y_2]} dx$$

Problem 1 of the Practice Exam

1. Explain in words and formulas how you would construct the general solution of $y'' + p(x)\,y' + q(x)\,y = g(x)$, given that $y_1(x)$ is a solution of $y'' + p(x)\,y' + q(x)\,y = 0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

- Step 1: Use Reduction of Order to find a second, independent, solution of the homogenous equation:

$$y_2 = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left(- \int^x p(s)\,ds \right) dx$$

- Step 2: Use Variations of Parameters to find a particular solution $y_p(x)$ of the inhomogeneous equation

$$y_p(x) = -y_1(x) \int \frac{y_2(x)\,g(x)}{W[y_1, y_2]} dx \quad + \quad y_2(x) \int \frac{y_1(x)\,g(x)}{W[y_1, y_2]} dx$$

- Step 3: The general solution of the inhomogeneous equation can now be constructed from y_1, y_2 and y_p :

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x)$$

Problem 2 of Practice Exam

2. Given that $y_1(x) = x^{-1}$ is one solution of $x^2y'' + xy' - y = 0$, use Reduction of Order to determine the general solution of this differential equation.

Problem 2 of Practice Exam

2. Given that $y_1(x) = x^{-1}$ is one solution of $x^2 y'' + xy' - y = 0$, use Reduction of Order to determine the general solution of this differential equation.

Dividing the given equation by x^2 puts it in standard form: $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$; and so

$$p(x) = \frac{1}{x}$$

Applying the Reduction of Order formula

$$y_2 = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left(-\int^x p(s)\,ds\right) dx = x^{-1} \int \frac{1}{(x^{-1})^2} \exp\Big[-$$

$$= x^{-1} \int x^2 \exp\left[-\ln|x|\right] dx = x^{-1} \int x^2 \left(\frac{1}{x}\right) dx$$

$$= x^{-1}\left(\frac{1}{2}x^2\right) = \frac{1}{2}x$$

Problem 2 of Practice Exam, Cont'd

With two independent solutions in hand, we can now write down the general solution:

$$y(x) = c_1 x^{-1} + c_2 \left(\frac{1}{2} x \right)$$

or

$$y(x) = c_1 x^{-1} + c_2 x$$

(since c_2 is just as arbitrary as $\frac{1}{2} c_2$).

Two Special Cases: Constant Coefficient and Euler-type Equations

	Constant Coefficients	Euler-type
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
Ansatz	$y(x) = e^{\lambda x}$	$y(x) = x^r$
Aux. Eq.	$a\lambda^2 + b\lambda + c = 0$	$ar^2 + (b-a)r + c = 0$
Case (i) 2 real roots	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$	$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$
Case (ii) 1 real root	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$	$y(x) = c_1 x^r + c_2 x^r \ln\lvert x\rvert$
Case (iii) 2 complex roots $\alpha \pm i\beta$	$y(x) = c_1 e^{\alpha x}\cos(\beta x)$ $+c_2 e^{\alpha x}\sin(\beta x)$	$y(x) = c_1 x^{\alpha}\cos(\beta \ln\lvert x\rvert)$ $+c_2 x^{\alpha}\sin(\beta \ln\lvert x\rvert)$

Problem 3 of the Practice Exam

3. Determine the general solution of the following differential equations.
(a) (5 pts) $y'' - 5y' + 6y = 0$
(b) (5 pt) $x^2y'' - 5xy' + 9y = 0$
(c) (5 pts) $y'' - 10y' + 25y = 0$
(d) (5 pts) $y'' + 2y' + 5y = 0.$
(e) (5 pts) $y'' + 4y' + 5y = 0$
(f) (5 pts) $x^2y'' - 5xy' + 13y = 0$

Problem 4 of Practice Exam

Given that $y_1(x) = x^{-1}$ and $y_2(x) = x^3$ are solutions of $x^2y'' - xy' - 3y = 0$:

Problem 4 of Practice Exam

Given that $y_1(x) = x^{-1}$ and $y_2(x) = x^3$ are solutions of $x^2y'' - xy' - 3y = 0$:
(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of $x^2y'' - xy' - 3y = 12x^2$.

Problem 4 of Practice Exam

Given that $y_1(x) = x^{-1}$ and $y_2(x) = x^3$ are solutions of $x^2y'' - xy' - 3y = 0$:
(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of $x^2y'' - xy' - 3y = 12x^2$.
(b) (10 pts) Find the solution of the differential equation in part (a) satisfying $y(1) = 0$, $y'(1) = 0$.

The Laplace Transform Method

The **Laplace transform** of a function $f(x)$ is

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x)\, dx \quad . \tag{1}$$

Theorem

(i) $\mathcal{L}\left[c_1 f_1 + c_2 f_2\right] = c_1 \mathcal{L}\left[f_1\right] + c_2 \mathcal{L}\left[f_2\right]$

(ii) $\mathcal{L}\left[\frac{df}{dx}\right] = s\mathcal{L}\left[f\right] - f\left(0\right)$

(iii) $\mathcal{L}\left[\frac{d^2 f}{dx^2}\right] = s^2 \mathcal{L}\left[f\right] - sf\left(0\right) - f'\left(0\right)$

Strategy for Solving ODEs via the Laplace Transform

Strategy for Solving ODEs via the Laplace Transform

1. Take the Laplace transform of the ODE, term-by-term, using the identities (i), (ii) and (iii) of the Theorem

Strategy for Solving ODEs via the Laplace Transform

1. Take the Laplace transform of the ODE, term-by-term, using the identities (i), (ii) and (iii) of the Theorem
2. Solve the resulting equation for $\mathcal{L}[y]$:

Strategy for Solving ODEs via the Laplace Transform

1. Take the Laplace transform of the ODE, term-by-term, using the identities (i), (ii) and (iii) of the Theorem
2. Solve the resulting equation for $\mathcal{L}[y]$:
3. Say $\mathcal{L}[y](s) = F(s)$. Figure out what function $f(x)$ has $F(s)$ as its Laplace transform (using a table of basic Laplace transforms)
4. Your solution will then be $y(x) = f(x)$.

Three Basic Cases for Inverting Laplace Transform of $F(s) = \frac{P(s)}{Q(s)}$

Three Basic Cases for Inverting Laplace Transform of $F(s) = \frac{P(s)}{Q(s)}$

- If $Q(s)$ can be written as a product $(s - a_2)^{m_1}(s - a_2)^{m_2} \cdots (s - a_k)^{m_k}$, try Partial Factions expansions.

Three Basic Cases for Inverting Laplace Transform of $F(s) = \frac{P(s)}{Q(s)}$

- If $Q(s)$ can be written as a product $(s-a_2)^{m_1}(s-a_2)^{m_2}\cdots(s-a_k)^{m_k}$, try Partial Factions expansions.
- If $Q(s)$ can be written in the form $(s-a)^2+b^2$, try to find coefficients A and B so that $F(s) = A\mathcal{L}\left[e^{ax}\cos(bx)\right] + B\mathcal{L}\left[e^{ax}\sin(bx)\right]$

Three Basic Cases for Inverting Laplace Transform of $F(s) = \frac{P(s)}{Q(s)}$

- If $Q(s)$ can be written as a product $(s-a_2)^{m_1}(s-a_2)^{m_2}\cdots(s-a_k)^{m_k}$, try Partial Factions expansions.
- If $Q(s)$ can be written in the form $(s-a)^2+b^2$, try to find coefficients A and B so that $F(s) = A\mathcal{L}\left[e^{ax}\cos(bx)\right] + B\mathcal{L}\left[e^{ax}\sin(bx)\right]$
- If $Q(s)$ can be written in the form $(s-a)^2-b^2$, try to find coefficients A and B so that $F(s) = A\mathcal{L}\left[e^{ax}\cosh(bx)\right] + B\mathcal{L}\left[e^{ax}\sinh(bx)\right]$

Problem 5 of Sample Exam

5. Invert the following Laplace Transforms (i.e find the function $f(t)$ with the given Laplace transform).

(a) (10 pts) $\mathcal{L}[f](s) = \dfrac{s-1}{s^2-4s}$

Problem 5 of Sample Exam

5. Invert the following Laplace Transforms (i.e find the function $f(t)$ with the given Laplace transform).

(a) (10 pts) $\mathcal{L}[f](s) = \dfrac{s-1}{s^2-4s}$

(b) (10 pts) $\mathcal{L}[f](s) = \dfrac{1}{s^2+2s+5}$ (Hint: try completing the square in the dominator)

Problem 5 of Sample Exam

5. Invert the following Laplace Transforms (i.e find the function $f(t)$ with the given Laplace transform).

(a) (10 pts) $\mathcal{L}[f](s) = \dfrac{s-1}{s^2-4s}$

(b) (10 pts) $\mathcal{L}[f](s) = \dfrac{1}{s^2+2s+5}$ (Hint: try completing the square in the dominator)

Problem 6 of the Sample Exam

6. Solve the following initial value problems using the Laplace transform method.

$$y'' + 3y' + 2y = 0 \qquad ; \qquad y(0) = 3 \quad , \quad y'(0) = -5$$