Math 2233 - Lecture 18

Agenda:

1. Solutions via Taylor Series
2. Using Power Series as Trial Solutions
3. Manipulating Power Series
4. Shifts of Summation Indices
5. Example: Solving an ODE via Power Series
6. Summary of the Power Series Method

We saw in Lecture 16 that, given a general initial value problem like

$$\begin{aligned} y'' + p(x)\,y' + q(x)\,y &= 0 \\ y(0) &= y_0 \\ y'(0) &= y_0' \end{aligned}$$

one can directly compute the Taylor expansion

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots$$

of the solution using only the initial conditions and the differential equation. For

$$\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_0' \\ y''(0) &= \left(-p(x)\,y'(x) - q(x)\,y(x)\right)\Big|_{x=0} \\ &= -p(0)\,y_0' - q(0)\,y_0 \end{aligned}$$

Taylor Series Solutions, Cont'd

To get values of the higher derivatives, $y'''(0), y^{(iv)}(0), \ldots$ we differentiate the differential equation and then evaluate at $x = 0$:

$$
\begin{aligned}
y'''(0) &= \left.\frac{d}{dx}\left[-\left[p(x)\right]y'(x) - q(x)\,y(x)\right]\right|_{x=0} \\
&= -p'(0)\,y'(0) - p(0)\,y''(0) - q'(0)\,y(0) - q(0)\,y'(0) \\
&= -p'(0)\,y_0' - p(0)\left(-p(0)\,y_0' - q(0)\,y_0\right) \\
&\quad -q'(0)\,y_0 - q(0)\,y_0' \\
y^{(iv)}(0) &= \left.\frac{d^2}{dx^2}\left[-\left[p(x)\right]y'(x) - q(x)\,y(x)\right]\right|_{x=0} \\
&= \text{something computable in terms of } y_0, y_0' \text{ and} \\
&\quad \text{the derivatives of } p(x) \text{ and } q(x) \text{ evaluated at } 0.
\end{aligned}
$$

and so on.

Solutions via Power Series

The direct calculation of the Taylor series of a solution gets very strenuous very quickly.
So this direct calculation is really only practical for finding approximate solutions (where you simply ignore the higher derivative terms, figuring that their contribution is small when x is close to 0).
So rather than calculate the numbers $y^{(n)}(0)$ directly by taking derivatives of the differential equation, we'll instead look of solutions that have the same functional form as a Taylor expansion

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{*}$$

i.e., a power series function and then figure out what the numbers a_n have to be in order for such a power series function, (*), to be a solution of the differential equation.

The basic idea here is just a generalization of what we did for constant coeffient and Euler-type equations

$$\begin{aligned} ay'' + by' + cy &= 0 \quad \Rightarrow \quad \text{look for solutions of the form } e^{\lambda x} \\ ax^2y'' + bxy' + cy &= 0 \quad \Rightarrow \quad \text{look for solution of the form } x^r \end{aligned}$$

except that now for differential equations of the very general form

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad \Rightarrow \quad \text{try } y(x) = \sum_{n=0}^{\infty} a_n x^n$$

If we can find numbers $a_0, a_1, a_2, \ldots$ such that the power series function

$$\sum_{n=0}^{\infty} a_n x^n$$

automatically satisfies the differential equation, then we will have calculated the complete Taylor expansion of the solution. (Because the Taylor series for a power series function is just the power series itself.)

1st Order Example

Here's a simple example of this idea (applied to a first order ODE to keep things really simple)

Consider the differential equation of the exponential function e^x:

$$y' = y \tag{*}$$

Write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{**}$$

as a trial solution and plug into the differential equation. We'll treat the right hand side of (*) as an "infinite polynomial". Thus

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad \text{(differentiating term-by-term)} \\ &= (0)\, a_0 x^{-1} + (1)\, a_1 x^0 + (2)\, a_2 x^1 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n \end{aligned}$$

1st Order Example, Cont'd

Plugging the power series expressions for y and y' into the differential equation (*) yields

$$\sum_{n=0}^{\infty}(n+1)\,a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n$$

Equating the coefficients of like powers of x, we find

$$(n+1)\,a_{n+1} = a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

$$\Rightarrow \quad a_{n+1} = \frac{a_n}{n+1} \qquad \text{for } n = 0, 1, 2, \ldots$$

This infinite set of equations are called the **Recursion Relations** for the problem. Using these recursion relations we can express each of the coefficients $a_1, a_2, a_3, \ldots$ in terms of a_0.

Applying the Recursion Relations $a_{n+1} = \frac{a_n}{n+1}$

$$\begin{aligned}
a_1 &= a_{0+1} = \frac{a_0}{1} = a_0 \\
a_2 &= a_{1+1} = \frac{a_1}{2} = \frac{a_0}{2 \cdot 1} \\
a_3 &= a_{2+1} = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2 \cdot 1} \\
&\vdots
\end{aligned}$$

and, continuing the following pattern emerges

$$a_n = \frac{a_0}{n!}$$

and so our solution will be

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n \\
&= a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n
\end{aligned}$$

1st Order Example, Cont'd

Thus, after posing the trial solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

we were able to construct the general solution (in terms of its Taylor series about $x = 0$):

$$y(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n$$

Manipulating Power Series

Our strategy for solving 2nd Order linear ODEs

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{1}$$

will be to similarly look for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad . \tag{2}$$

To carry out this plan, we are going to need to know

(i) How to differentiate power series expressions

(ii) How to multiple power series by (typically polynomial) functions $p(x)$ and $q(x)$

(iii) How to add power series expressions

(iv) How to extract conditions on the coefficients a_n from power series equations

Manipulating Power Series, Cont'd

These questions are answered quite simply by the following theorem

Theorem

So long as the power series converge, they behave like infinite polynomials:

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{i}$$

$$\left(c_0 + c_1 x + \cdots\right)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c_0 a_n x^n + \sum_{n=0}^{\infty} c_1 a_n x^{n+1} + \cdots \tag{ii}$$

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(a_n + b_n\right) x^n \tag{iii}$$

$$0 = \sum_{n=0}^{\infty} a_n x^n \ \textit{for all } x \quad \Rightarrow \quad a_n = 0 \ \ \textit{for all } n \tag{iv}$$

Shifts of Summation Indices

The one difficulty one faces when manipulating power series expressions is that the simple rule for adding power series

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n \qquad \text{(iii)}$$

only works exactly as written.
Thus, if $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$y' + y = \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \neq \sum_{n=0}^{\infty} (n a_n + a_n) x^n$$

because one **cannot** group together the coefficients na_n and a_n that correspond to **different powers of** x (respectively, x^{n-1} and x^n). The Rule (iii) requires that we combine the coefficients of the same power of x to get the sum of two power series.

Shifts of Summation Indices, Cont'd

However, there is a simple operation that allows us to add power series where the summation indexes n don't necessarily have to be the same as the power of x in the corresponding series.

Theorem (Shift of Summation Index)

$$\sum_{n=0} a_n x^{n+k} = \sum_{n=k} a_{n-k} x^n$$

Proof. Expanding the left hand side of the above identity, we get

$$\sum_{n=0} a_n x^{n+k} = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \cdots$$

Doing the same thing on the right we get

$$\begin{aligned} \sum_{n=k} a_{n-k} x^n &= a_{k-k} x^k + a_{(k+1)-k} x^{k+1} + a_{(k+2)-k} x^{k+2} \cdots \\ &= a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \cdots \end{aligned}$$

Thus, the two expansions agree with one another.

Shift of Summation Indices, Cont'd

Here is a restatement (and slight generalization) of this basic rule:

Definition

A power series is in **standard form** whenever the powers of x that occur coincide with the summation index n. That is to say, the power series is exactly of the simple form

$$\sum_{n=0}^{\infty} a_n x^n$$

A power series that is not in standard form, say,

$$\sum_{n=n_0}^{\infty} a_n x^{n \pm k}$$

can be rewritten in standard form by replacing the initital value n_0 of n by $n_0 \pm k$ and then replacing n by $n \mp k$ everywhere else in the original power series

Shift of Summation Indices: Summary

Thus,

$$\boxed{\sum_{n=n_0}^{\infty} a_n x^{n\pm k} \quad \xrightarrow[]{\substack{n_0 \to n_0 \pm k \\ n \to n \mp k}} \quad \sum_{n=n_0\pm k}^{\infty} a_{n\mp k} x^n}$$

or,

- **If you need to shift the n in x^{n+k} down by k (to get x^n)**
  - Replace n by $n-k$ everywhere to the right of the summation sign
  - Also shift the starting value of n by k, **but in the opposite direction**
- **If you need to shift the n in x^{n-k} up by k (to get x^n)**
  - replace n by $n+k$ everywhere to the right of the summation sign
  - Also shift the starting value of n by k **but in the opposite direction**

Shift of Summation Indices: Example

Put the following double derivative term in standard form

$$y'' = \sum_{n=0}^{\infty} (n)(n-1)\, a_n x^{n-2}$$

We need to carry out a summation shift so that $x^{n-2} \to x^n$. To do this clearly, set

$$m = n - 2 \quad \Rightarrow \quad n = m + 2$$

then the initial value of m will be

$$m_0 = n_0 - 2 = 0 - 2 = -2$$

Now make these substitutions into the original power series expression

$$\begin{aligned} \sum_{n=0}^{\infty} (n)(n-1)\, a_n x^{n-2} &= \sum_{m=-2}^{\infty} (m+2)(m+2-1)\, a_{m+2} x^{m+2-2} \\ &= \sum_{m=-2} (m+2)(m+1)\, a_{m+2} x^m \end{aligned}$$

Shift of Summation Indices Example, Cont'd

This not quite yet in standard form since the power series begins at $m = -2$ rather than $m = 0$. However, we can always "peel off" the initial terms of a power series and deal with them separately. In the case at hand, we have

$$\begin{aligned}
\sum_{m=-2} (m+2)(m+1)\, a_{m+2} x^m &= (-2+2)(-2+1)\, a_{-2+2} x^{-2} \\
&\quad + (-1+2)(-1+1)\, a_{-1+2} x^{-1} \\
&\quad + \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} x^m \\
&= 0 + 0 + \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} x^m \\
&= \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} x^m
\end{aligned}$$

N.B. This last series is in the standard form $\sum_{n=0}^{\infty} A_n x^n$ with

$$A_n \equiv (n+2)(n+1)\, a_n$$

Application to Solving ODEs

Now consider the differential equation

$$y'' - xy' - y = 0$$

We're going to find a power series solution of this differential equation. We begin by setting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and plugging this trial solution into the differential equation. We have

$$\begin{aligned}
y'' &= \sum_{n=0}^{\infty} (n)(n-1) a_n x^{n-2} && \text{(by direct differentiation)} \\
&= \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n && \text{(after a shift of summation index)} \\
&= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n && \text{(after peeling off initial terms)}
\end{aligned}$$

Example: Power Series Solutions of ODEs

The next term we need to deal with is xy'

$$\begin{aligned} xy' &= x\sum_{n=0}^{\infty} na_n x^{n-1} \\ &= \sum_{n=0} na_n x^{n-1+1} \qquad \text{(bringing the factor } x \text{ through the summatio} \\ &= \sum_{n=0}^{\infty} na_n x^n \end{aligned}$$

We now have each term of the differential equation expressed as a **power series in standard form**.

Example: Power Series Solutions of ODEs

Thus, if $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution of

$$0 = y'' - xy' - y$$

we must have

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

Since each power series on the right is in standard form, we can combine the terms into a single power series by collecting the total coefficient of each x^n

$$0 = \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - n a_n - a_n\right] x^n$$

Example: Power Series Solutions of ODEs, Cont'd

Since this last power series must equal 0 for all x, all of its coefficients must separately vanish:

$$0 = (n+2)(n+1)\,a_{n+2} - (n+1)\,a_n \qquad , \qquad n = 0, 1, 2, \ldots$$

or

$$a_{n+2} = \frac{(n+1)\,a_n}{(n+2)(n+1)} = \frac{a_n}{n+2} \qquad , \qquad n = 0, 1, 2, \ldots \quad (RR_n)$$

The equations RR_n are called the **Recursion Relations** for the problem. I'll now show you how the recursion relations can be used to write the **general solution** of the original differential equation.

Example: Power Series Solutions of ODEs, Cont'd

Set

$$\begin{aligned} a_0 &= c_1 \\ a_1 &= c_2 \end{aligned}$$

where c_1 and c_2 are arbitrary constants
Then

$$\begin{aligned}
RR_0 \quad &\Rightarrow \quad a_2 = a_{0+2} = \frac{a_0}{0+2} = \frac{1}{2} c_1 \\
RR_1 \quad &\Rightarrow \quad a_3 = a_{1+2} = \frac{a_1}{1+2} = \frac{1}{3} c_2 \\
RR_2 \quad &\Rightarrow \quad a_4 = a_{2+2} = \frac{a_2}{2+2} = \frac{1}{(4)(2)} c_1 \\
RR_3 \quad &\Rightarrow \quad a_5 = a_{3+2} = \frac{a_3}{3+2} = \frac{1}{(5)(3)} c_2 \\
&\vdots
\end{aligned}$$

Example: Power Series Solutions of ODEs, Cont'd

It should be clear that we can continue to compute as many of the remaining coefficients $a_6, a_7, \ldots$ as we want.
But let's instead start to write down the solution to see what it looks like

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\
&= c_1 + c_2 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 + \frac{1}{8} c_1 x^4 + \frac{1}{15} c_2 x^5 + \cdots \\
&= c_1 \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \cdots\right) + c_2 \left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots\right) \\
&= c_1 y_1(x) + c_2 y_2(x)
\end{aligned}
$$

Example: Power Series Solutions of ODEs, Cont'd

In that last equation, we have expressed our solution as a linear combination of the two independent solutions

$$
\begin{aligned}
y_1(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots \\
y_2(x) &= x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots
\end{aligned}
$$

In fact, y_1 and y_2 are the following two special solutions with particularly simple initial conditions at $x = 0$.

$y_1(x) =$ the unique solution of the ODE satisfying

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

$y_2(x) =$ the unique solution satisfying

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

Summary: The Power Series Method

Goal: Find a solution of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (*)$$

1. Substitute (*) into each term of the ODE and then manipulate the resulting power series expression until it is in standard form.
2. Power series in standard form are readily added together, and so after Step 1. we'll see that the differential equation implies an equation of the form

$$0 = \sum_{n=0}^{\infty} A_n(n, a_{n+2}, \dots, a_0)\, x^n = 0$$

which will in turn imply an infinite set of equations $A_n(n, a_{n+2}, \dots) = 0$

Summary: The Power Series Method, Cont'd

3. Solve these equations for the highest coefficient that appears in them, say its a_{n+2}:

   $a_{n+2} =$ some function of n and the lower coefficients $a_{n-1}, \ldots, a_0$

4. The resulting equations will be naturally organized so that you can systematically compute the higher coefficients $a_2, a_3, \ldots$ in terms of the first two a_0 and a_1. Set $a_0 = c_1$ and $a_1 = c_2$ and then compute as many a_n as you need.

5 Collect together the terms with c_1 as a factor as $c_1 y_1(x)$ and those with c_2 as a factor as $c_2 y_2(x)$/ Then you'll be able to express the general solution of the ODE in the usual form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$