Math 2233 - Lecture 18

Agenda:

1. Solutions via Taylor Series
2. Using Power Series as Trial Solutions
3. Manipulating Power Series
4. Shifts of Summation Indices
5. Example: Solving an ODE via Power Series
6. Summary of the Power Series Method

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i.e., a power series function and then figure out what the numbers a_n have to be in order for such a power series function, (*), to be a solution of the differential equation.

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This infinite set of equations are called the **Recursion Relations** for the problem. Using these recursion relations we can express each of the coefficients $a_1, a_2, a_3, \ldots$ in terms of a_0.

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we were able to construct the general solution (in terms of its Taylor series about $x = 0$):

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Manipulating Power Series

Our strategy for solving 2nd Order linear ODEs

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{1}$$

will be to similarly look for solutions of the form

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(iv) How to extract conditions on the coefficients a_n from power series equations

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$$(c_0 + c_1 x + \cdots)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c_0 a_n x^n + \sum_{n=0}^{\infty} c_1 a_n x^{n+1} + \cdots \qquad \text{(ii)}$$

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$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n \qquad \text{(iii)}$$

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The one difficulty one faces when manipulating power series expressions is that the simple rule for adding power series

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Thus, the two expansions agree with one another.

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can be rewritten in standard form by replacing the initital value n_0 of n by $n_0 \pm k$ and then replacing n by $n \mp k$ everywhere else in the original power series

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Put the following double derivative term in standard form

$$y'' = \sum_{n=0}^{\infty} (n)\,(n-1)\,a_n x^{n-2}$$

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$$\begin{aligned} xy' &= x\sum_{n=0}^{\infty} na_n x^{n-1} \\ &= \sum_{n=0} na_n x^{n-1+1} \qquad \text{(bringing the factor } x \text{ through the summatio} \\ &= \sum_{n=0}^{\infty} na_n x^n \end{aligned}$$

We now have each term of the differential equation expressed as a **power series in standard form**.

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$$0 = \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - n a_n - a_n\right] x^n$$

Example: Power Series Solutions of ODEs, Cont'd

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The equations RR_n are called the **Recursion Relations** for the problem.

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The equations RR_n are called the **Recursion Relations** for the problem. I'll now show you how the recursion relations can be used to write the **general solution** of the original differential equation.

Example: Power Series Solutions of ODEs, Cont'd

Set

$$\begin{aligned} a_0 &= c_1 \\ a_1 &= c_2 \end{aligned}$$

where c_1 and c_2 are arbitrary constants

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RR_2 \quad &\Rightarrow \quad a_4 = a_{2+2} = \frac{a_2}{2+2} = \frac{1}{(4)(2)}c_1 \\
RR_3 \quad &\Rightarrow \quad a_5 = a_{3+2} = \frac{a_3}{3+2} = \frac{1}{(5)(3)}c_2 \\
&\vdots
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$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \end{aligned}$$

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&= c_1 + c_2 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 + \frac{1}{8} c_1 x^4 + \frac{1}{15} c_2 x^5 + \cdots
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&= c_1 \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \cdots\right) + c_2 \left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots\right)
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&= c_1 \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \cdots \right) + c_2 \left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots \right) \\
&= c_1 y_1(x) + c_2 y_2(x)
\end{aligned}$$

Example: Power Series Solutions of ODEs, Cont'd

In that last equation, we have expressed our solution as a linear combination of the two independent solutions

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots \\ y_2(x) &= x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots \end{aligned}$$

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$y_2(x) =$ the unique solution satisfying

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Summary: The Power Series Method

Goal: Find a solution of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (*)$$

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$$0 = \sum_{n=0}^{\infty} A_n\left(n, a_{n+2}, \ldots, a_0\right) x^n = 0$$

which will in turn imply an infinite set of equations $A_n\left(n, a_{n+2}, \ldots\right) = 0$

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4. The resulting equations will be naturally organized so that you can systematically compute the higher coefficients $a_2, a_3, \ldots$ in terms of the first two a_0 and a_1. Set $a_0 = c_1$ and $a_1 = c_2$ and then compute as many a_n as you need.

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5 Collect together the terms with c_1 as a factor as $c_1 y_1(x)$ and those with c_2 as a factor as $c_2 y_2(x)$/ Then you'll be able to express the general solution of the ODE in the usual form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$