Math 2233 - Lecture 19

Agenda:

1. Solutions via Power Series
2. Power Series Manipulations
3. Example: Putting a Power Series in Standard Form
4. Example: The Power Series Method without Initial Conditions
5. Example: The Power Series Method with Initial Conditions
6. Power Series about $x_0 \neq 0$
7. Example: A Power Series Solution with Initial Conditions at $x = 1$
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Power Series

A function of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is called a **power series**.
Such functions behave like infinite polynomials.
In fact, polynomial functions are just a special case of power series functions

$$\begin{aligned} x^2 + 2x + 3 &= 3 + 2x + x^2 + 0 + 0 + \cdots \\ &= \sum_{n=0}^{\infty} a_n x^n \quad \text{when} \quad a_n = \begin{cases} 3 \text{ if } n = 0; \\ 2 \text{ if } n = 1 \\ 1 \text{ if } n = 2 \\ 0 \text{ if } n \geq 3 \end{cases} \end{aligned}$$

The idea we have been pursuing is that power series can be used as "trial solutions" to differential equations of the form

$$y'' + p(x)y' + q(x)\,y = 0 \tag{1}$$

That is, we propose that there are solutions of (1) of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

and we then try to use the differential equation to figure out the exactly what choice of coefficients $a_0, a_1, \ldots,$ will ensure that (2) is actually a solution of (1).

Power Series Manipulations

(i) Differentiating power series

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{i}$$

(ii) Multiplying power series by polynomial functions

$$(c_0 + c_1 x + \cdots)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c_0 a_n x^n + \sum_{n=0}^{\infty} c_1 a_n x^{n+1} + \cdots \tag{ii}$$

(iii) Adding power series expressions

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n \tag{iii}$$

(iv) Extracting conditions on the coefficients a_n from power series equations

$$0 = \sum_{n=0}^{\infty} a_n x^n \text{ for all } x \quad \Rightarrow \quad a_n = 0 \ \text{ for all } n \tag{iv}$$

However, as we saw in Lecture 18, there are multiple ways of presenting the same power series using summation notation:

Definition

A power series is in **standard form** is a power series expression where the power of x coincides with the summation index.

E.g.,

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

In the equality above, the power series on the right is not in standard form, but the power series on the right is in standard form.

This distinction is important, because the rule for adding power series

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n \tag{iii}$$

only works when both power series are in standard form.

Meanwhile operations like differentiation

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{i}$$

or multiplying a power series by a function

$$(c_0 + c_1 x + \cdots)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c_0 a_n x^n + \sum_{n=0}^{\infty} c_1 a_n x^{n+1} + \cdots \tag{ii}$$

tend to produce power series functions that are not in standard form.

Shifts of Summation Indices

Therefore we often have to rewrite power series in standard form prior to adding it to other power series. This manipulation is called a **shift of summation index**. Here are the rules

$$\begin{aligned}
\sum_{n=n_0}^{\infty} A_n x^{n+k} &= \sum_{n=n_0+k}^{\infty} A_{n-k} x^n \\
\sum_{n=n_0}^{\infty} A_n x^{n-k} &= \sum_{n=n_0-k}^{\infty} A_{n+k} x^n
\end{aligned}$$

Note how the starting value of n always get shifted in the opposite direction.

Shift of Summation Index Example

Suppose

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0} a_n x^{n+1} = 0$$

What does this tell us about the coefficients a_n?

Let's first put each power series expression on the right in standard form

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \xrightarrow{\quad n \to n+2 \quad} \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\sum_{n=0} a_n x^{n+1} \xrightarrow{\quad n \to n-1 \quad} \sum_{n=1}^{\infty} a_{n-1}x^n$$

So

$$
\begin{aligned}
0 &= \sum_{n=-2}^{\infty} (n+2)(n+1)\, a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n \\
&= (0)(-1)\, a_0 x^{-2} + (1)(0)\, a_1 x^{-1} + (2)(1)\, a_2 x^0 \\
&\quad + \sum_{n=1}^{\infty} (n+2)(n+1)\, a_{n+2} x^n \\
&\quad - \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}
$$

or

$$
0 = 0 + 0 + 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1} \right] x^n
$$

We thus have

$$0 = 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1} \right] x^n$$

If a power series equals 0 for all x, then all of its coefficients have to equal 0. Thus, from the expression above, we see on the right

$$0 = \text{coefficient of } x^0 = 2a_2 \quad \Rightarrow \quad a_2 = 0$$

and, for $n = 1, 2, 3, \ldots$ (i.e., for each n in the summation)

$$0 = (n+2)(n+1)\, a_{n+2} - a_{n-1} \quad \Rightarrow \quad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

So we have

$$\begin{aligned} a_2 &= 0 \\ a_{n+2} &= \frac{a_{n-1}}{(n+2)(n+1)} \qquad , \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Example: Solving an IVP via Power Series

Consider the following initial value problem:

$$\begin{aligned} (1-x)y'' - y &= 0 \\ y(0) &= 2 \\ y'(0) &= 1 \end{aligned}$$

We look for solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
The initial conditions tell us that

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 1 \end{aligned}$$

We now need to figure out values for $a_2, a_3, \ldots$

Let's look at the first term of the differential equation

$$\begin{aligned}
(1-x)\,y'' &= (1-x)\sum_{n=0}^{\infty} n\,(n-1)\,a_n x^{n-2} \\
&= \sum_{n=0}^{\infty} n\,(n-1)\,a_n x^{n-2} - \sum_{n=0}^{\infty} n\,(n-1)\,a_n x^{n-1} \\
&= \sum_{n=-2}^{\infty} (n+2)\,(n+1)\,a_{n+2} x^{n} - \sum_{n=-1}^{\infty} (n+1)\,(n)\,a_{n+1} x^{n} \\
&= 0 + 0 + \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2} x^{n} \\
&\quad -0 + \sum_{n=0}^{\infty} (n+1)\,n a_{n+1} x^{n} \\
&= \sum_{n=0}^{\infty} \left[(n+2)\,(n+1)\,a_{n+2} - (n+1)\,n a_{n+1}\right] x^{n}
\end{aligned}$$

Good, we have $(x-1)y''$ expressed as a power series in standard form. We can now add it to the term $-y = \sum_{n=0}^{\infty} -a_n x^n$ (which is already in standard form)

$$
\begin{aligned}
0 &= (1-x)\,y'' - y \\
&= \sum_{n=0}^{\infty} \left[(n+2)\,(n+1)\,a_{n+2} - (n+1)\,na_{n+1}\right] x^n + \sum_{n=0}^{\infty} -a_n x^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)\,(n+1)\,a_{n+2} - (n+1)\,a_{n+1} - a_n\right] x^n
\end{aligned}
$$

and so we must have

$$\left[(n+2)\,(n+1)\,a_{n+2} - (n+1)\,na_{n+1} - a_n\right] = 0 \qquad , \quad n = 0, 1, 2, \ldots$$

or

$$a_{n+2} = \frac{n\,(n+1)\,a_{n+1} + a_n}{(n+2)\,(n+1)} \qquad , \quad n = 0, 1, 2, \ldots$$

We can now begin calculating the higher coefficients using the above recursion relations and the initial values $a_0 = 2$ and $a_1 = 1$

$$
\begin{aligned}
a_2 &= a_{0+2} = \frac{(0)(1)a_1 + a_0}{(2)(1)} = \frac{a_0}{2} = 1 \\
a_3 &= a_{1+2} = \frac{(1)(2)a_2 + a_1}{(3)(2)} = \frac{2a_2 + a_1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{1}{2}
\end{aligned}
$$

We can continue computing $a_4, a_5, \ldots$ in the same way. But let's instead write down at least the beginning of the power series solution

$$
\begin{aligned}
y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= 2 + x + x^2 + \frac{1}{2}x^3 + \cdots
\end{aligned}
$$

Initial Values at a Point $x_0 \neq 0$

Consider the following initial value problem

$$\begin{aligned} xy'' - y &= 0 \\ y(1) &= 1 \\ y'(1) &= 2 \end{aligned}$$

If we pose a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

In this case, the initial value conditions say

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} a_n (1)^n = a_0 + a_1 + \cdots \\ 2 &= \sum_{n=0}^{\infty} n a_n (1)^n = a_1 + 2a_2 + \cdots \end{aligned}$$

and so we can no longer use the initial conditions to get values for a_0 and a_1.

Power Series about $x_0 \neq 0$

Rather, one has to instead use a power series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \tag{3}$$

This is still a power series, but now is a **power series about** $x = 1$. It corresponds to taking a Taylor expansion about $x = 1$ rather than about $x = 0$.

So let us adopt (3) as our trial solution. The initial conditions applied to (3) say

$$\begin{aligned} 1 &= y(1) = \sum_{n=0}^{\infty} a_n (1-1)^n \quad \Rightarrow \quad a_0 = 1 \\ 2 &= y'(1) = \sum_{n=0}^{\infty} n a_n (1-1)^{n-1} \quad \Rightarrow \quad a_1 = 2 \end{aligned}$$

So, after the change in the expansion point, the initial conditions again give us the first two coefficients a_0 and a_1.

Let's now compute xy'' using the power series (3) about $x = 1$

$$xy'' = x \sum_{n=0}^{\infty} n(n-1) a_n (x-a)^{n-2}$$

Here we have another issue, we can't just bring the factor of x through the sum and adjust the power of $(x-a)^n$

$$x(x-a)^{n-2} \neq (x-a)^{n-1}$$

What you have to do is replace the factor x in front by its Taylor expansion about $x = 1$.

$$f(x) = x \quad \Rightarrow \quad f(1) = 1 \quad , \quad f'(1) = 1 \quad , \quad f^{(n)}(1) = 0 \text{ for } n = 2, 3,$$

Thus,

$$\begin{aligned} x &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \cdots \\ &= 1 + (x-1) \end{aligned}$$

And so

$$
\begin{aligned}
xy'' &= (1+(x-1))\sum_{n=0}^{\infty} n(n-1)\,a_n\,(x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} n(n-1)\,a_n\,(x-1)^{n-2} + (x-1)\sum_{n=0}^{\infty} n(n-1)\,a_n\,(x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} n(n-1)\,a_n\,(x-1)^{n-2} + \sum_{n=0}^{\infty} n(n-1)\,a_n\,(x-1)^{n-1} \\
&= \sum_{n=-2}^{\infty} (n+2)(n+1)\,a_{n+2}\,(x-1)^{n} + \sum_{n=-1}^{\infty} (n+1)(n)\,a_{n+1}\,(x-1)^{n} \\
&= 0+0+\sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\,(x-1)^{n} \\
&\quad +0+\sum_{n=0}^{\infty} n(n+1)\,a_{n+1}\,(x-1)^{n} \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\,a_{n+2} + n(n+1)\,a_{n+1}\right](x-1)^{n}
\end{aligned}
$$

Our expression for xy'' is now in standard form and so we can combine it with the other term in the differential equation (which is already in standard form)

$$\begin{aligned} 0 &= xy'' - y \\ &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + n(n+1)\, a_{n+1} \right] (x-1)^n \\ &\quad - \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + n(n+1)\, a_{n+1} - a_n \right] (x-1)^n \end{aligned}$$

Thus,

$$a_{n+2} = \frac{-n(n+1)\, a_{n+1} + a_n}{(n+2)(n+1)}$$

These recursion relations imply

$$
\begin{aligned}
a_2 &= a_{0+2} = \frac{-(0)(1)a_1 + a_0}{(2)(1)} = \frac{a_0}{2} = \frac{1}{2} \\
a_3 &= a_{1+2} = \frac{-(1)(2)a_2 + a_1}{(3)(2)} = -\frac{1}{3}a_2 + \frac{1}{6}a_1 = -\frac{1}{6} + \frac{2}{6} = \frac{1}{6}
\end{aligned}
$$

and so to $\mathcal{O}\left((x-1)^3\right)$

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots \\
&= 1 + 2(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \cdots
\end{aligned}
$$

Summary: Power Series solutions about $x = x_0$

Use the trial solution

$$y(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n \tag{4}$$

whenever

- you have initial conditions defined at a point $x = x_0$; or
- you want to find approximate solutions that are accurate near $x = x_0$

For power series of the form (4):

$$\begin{aligned} a_0 &= y\left(x_0\right) \\ a_1 &= y'\left(x_0\right) \end{aligned}$$

One thing to be careful of:
If you a function $p(x)$ is multiplying a power series like (4), you need to replace $p(x)$ by its Taylor series about x_0, before bringing it through the summation.

Rule for Multiplying $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ by a function $f(x)$

Compute the Taylor expansion of $f(x)$ about $x = x_0$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{n!}(x - x_0)^2 + \cdots$$

Then

$$\begin{aligned} f(x)\sum_{n=0}^{\infty} a_n (x - x_0)^n &= f(x_0)\sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &\quad + f'(x_0)(x - x_0)\sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &\quad + \frac{1}{2} f''(x_0)(x - x_0)^2 \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &\quad + \cdots \end{aligned}$$

Rule for Multiplying $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ by a function $f(x)$, Cont'd

or

$$\begin{aligned} f(x) \sum_{n=0}^{\infty} a_n (x - x_0)^n &= \sum_{n=0}^{\infty} f(x_0)\, a_n (x - x_0)^n \\ &\quad + \sum_{n=0}^{\infty} f'(x_0)\, a_n (x - x_0)^{n+1} \\ &\quad + \sum_{n=0}^{\infty} \frac{1}{2} f''(x_0)\, a_n (x - x_0)^{n+2} \\ &\quad + \cdots \end{aligned}$$

Example: Multiplication by a function

Write

$$x^2 \sum_{n=0}^{\infty} a_n (x-1)^n$$

as a power series about $x = 1$.

If

$$f(x) = x^2$$

then

$$\begin{aligned} f(1) &= 1 \\ f'(1) &= 2x|_{x=1} = 2 \\ f''(1) &= 2|_{x=1} = 2 \\ f^{(n)}(1) &= 0 \text{ if } n > 2 \end{aligned}$$

and so

$$\begin{aligned} x^2 &= f(x) = f(0) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots \\ &= 1 + 2(x-1) + \frac{2}{2}(x-1)^2 \end{aligned}$$

and so

$$\begin{aligned}
x^2 \sum_{n=0}^{\infty} a_n (x-1)^n &= \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} a_n (x-1)^n + 2(x-1) \sum_{n=0}^{\infty} a_n (x-1)^n \\
&\quad + (x-1)^2 \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^{n+1} \\
&\quad + \sum_{n=0}^{\infty} a_n (x-1)^{n+2}
\end{aligned}$$

Now we'll have to shift some summation indices and write some of initial terms separately

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+2} \\
&= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} 2a_{n-1} (x-1)^n + \sum_{n=2}^{\infty} a_{n-2} (x-1)^n \\
&= a_0 + a_1 (x-1) + \sum_{n=2}^{\infty} a_n (x-1)^n \\
&\quad +2a_0 (x-1) \sum_{n=2}^{\infty} a_{n-1} (x-1)^n \\
&\quad + \sum_{n=2}^{\infty} a_n (x-1)^n \\
&= a_0 + (a_1 + 2a_0)(x-1) + \sum_{n=2}^{\infty} [a_n + 2a_{n-1} + a_{n-2}] (x-1)^n
\end{aligned}
$$

Example

Example:

$$\begin{aligned} y'' - xy' - y &= 0 \\ y(1) &= 1 \\ y'(1) &= 2 \end{aligned}$$

Since the initial conditions are defined at $x = 1$, our trial solution will be of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

For this $y(x)$

$$\begin{aligned} y(1) &= 1 \quad \Rightarrow \quad a_0 = 1 \\ y'(1) &= 2 \quad \Rightarrow \quad a_1 = 2 \end{aligned}$$

Example, Cont'd

Next, we compute power series expressions in standard form representing each term in the differential equation:

$$\begin{aligned} y'' &= \sum_{n=0}^{\infty} n(n-1)\,a_n\,(x-1)^{n-2} \\ &= 0+0+\sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}\,(x-1)^n \end{aligned}$$

$$\begin{aligned} -xy' &= (-1-(x-1))\sum_{n=0}^{\infty} na_n\,(x-1)^{n-1} \\ &= \sum_{n=0}^{\infty} -na_n\,(x-1)^{n-1} + \sum_{n=0}^{\infty} -na_n\,(x-1)^n \\ &= 0+\sum_{n=0}^{\infty} -(n+1)\,a_{n+1}\,(x-1)^n + +\sum_{n=0}^{\infty} -na_n\,(x-1)^n \end{aligned}$$

Example, Cont'd

$$qq - xy' \;=\; \sum_{n=0}^{\infty} \left[-\left(n+1\right) a_{n+1} - na_n\right] \left(x-1\right)^n$$

and, of course,

$$-y = \sum_{n=0}^{\infty} -a_n \left(x-1\right)^n$$

Example, Cont'd

Thus,

$$\begin{aligned}
0 &= y'' - xy' - y \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} (x-1)^n \\
&\quad + \sum_{n=0}^{\infty} \left[-(n+1)\, a_{n+1} - n a_n \right] (x-1)^n \\
&\quad + \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - (n+1)\, a_{n+1} - n a_n + a_n \right] (x-1)^n
\end{aligned}$$

and so

$$a_{n+2} = \frac{(n+1)\, a_{n+1} + n a_n - a_n}{(n+2)(n+1)} = \frac{(n+1)\, a_{n+1} + (n-1)\, a_n}{(n+2)(n+1)}$$

Example, Cont'd

Thus,

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 2\\
a_2 &= a_{0+2} = \frac{(1)\,a_1 - a_0}{(2)\,(1)} = \frac{1}{2}a_1 - \frac{1}{2}a_0 = 1 - \frac{1}{2} = \frac{1}{2}\\
a_3 &= a_{1+2} = \frac{(2)\,a_2 + (0)\,a_1}{(2)\,(2)} = \frac{a_2}{2} = \frac{1}{4}
\end{aligned}$$

and so

$$\begin{aligned}
y\,(x) &= a_0 + a_1\,(x-1) + a_2\,(x-1)^2 + a_3\,(x-1)^3 + \cdots\\
&= 1 + 2\,(x-1) + \frac{1}{2}\,(x-1)^2 + \frac{1}{4}\,(x-1)^3 + \cdots
\end{aligned}$$