Math 2233 - Lecture 19

Agenda:

1. Solutions via Power Series
2. Power Series Manipulations
3. Example: Putting a Power Series in Standard Form
4. Example: The Power Series Method without Initial Conditions
5. Example: The Power Series Method with Initial Conditions
6. Power Series about $x_0 \neq 0$
7. Example: A Power Series Solution with Initial Conditions at $x = 1$
8. Multiplying $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ by a function

Power Series

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$$\begin{aligned} x^2 + 2x + 3 &= 3 + 2x + x^2 + 0 + 0 + \cdots \\ &= \sum_{n=0}^{\infty} a_n x^n \quad \text{when} \quad a_n = \begin{cases} 3 \text{ if } n = 0; \\ 2 \text{ if } n = 1 \\ 1 \text{ if } n = 2 \\ 0 \text{ if } n \geq 3 \end{cases} \end{aligned}$$

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and we then try to use the differential equation to figure out the exactly what choice of coefficients $a_0, a_1, \ldots,$ will ensure that (2) is actually a solution of (1).

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(iv) Extracting conditions on the coefficients a_n from power series equations

$$0 = \sum_{n=0}^{\infty} a_n x^n \text{ for all } x \quad \Rightarrow \quad a_n = 0 \;\text{ for all } n \qquad \text{(iv)}$$

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This distinction is important, because the rule for adding power series

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n \qquad \text{(iii)}$$

only works when both power series are in standard form.

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tend to produce power series functions that are not in standard form.

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Note how the starting value of n always get shifted in the opposite direction.

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and so we must have

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\left[(n+2)(n+1)\,a_{n+2} - (n+1)\,na_{n+1} - a_n\right] = 0 \qquad , \quad n = 0, 1, 2, \ldots
$$

Good, we have $(x-1)y''$ expressed as a power series in standard form. We can now add it to the term $-y = \sum_{n=0}^{\infty} -a_n x^n$ (which is already in standard form)

$$\begin{aligned}
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or

$$a_{n+2} = \frac{n(n+1)\,a_{n+1} + a_n}{(n+2)(n+1)} \qquad , \quad n = 0, 1, 2, \ldots$$

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So, after the change in the expansion point, the initial conditions again give us the first two coefficients a_0 and a_1.

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For power series of the form (4):

$$\begin{aligned} a_0 &= y\left(x_0\right) \\ a_1 &= y'\left(x_0\right) \end{aligned}$$

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Use the trial solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{4}$$

whenever

- you have initial conditions defined at a point $x = x_0$; or
- you want to find approximate solutions that are accurate near $x = x_0$

For power series of the form (4):

$$\begin{aligned} a_0 &= y(x_0) \\ a_1 &= y'(x_0) \end{aligned}$$

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One thing to be careful of:
If you a function $p(x)$ is multiplying a power series like (4), you need to replace $p(x)$ by its Taylor series about x_0, before bringing it through the summation.

Rule for Multiplying $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ by a function $f(x)$

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Compute the Taylor expansion of $f(x)$ about $x = x_0$:

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$$\begin{aligned} f(x) \sum_{n=0}^{\infty} a_n (x - x_0)^n &= f(x_0) \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &\quad + f'(x_0)(x - x_0) \sum_{n=0}^{\infty} a_n (x - x_0)^n \end{aligned}$$

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Rule for Multiplying $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ by a function $f(x)$, Cont'd

or

$$
\begin{aligned}
f(x) \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad = \quad & \sum_{n=0}^{\infty} f(x_0)\, a_n (x - x_0)^n \\
& + \sum_{n=0}^{\infty} f'(x_0)\, a_n (x - x_0)^{n+1} \\
& + \sum_{n=0}^{\infty} \frac{1}{2} f''(x_0)\, a_n (x - x_0)^{n+2} \\
& + \cdots
\end{aligned}
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Example: Multiplication by a function

Write

$$x^2 \sum_{n=0}^{\infty} a_n (x-1)^n$$

as a power series about $x = 1$.

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and so

$$\begin{aligned} x^2 &= f(x) = f(0) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots \\ &= 1 + 2(x-1) + \frac{2}{2}(x-1)^2 \end{aligned}$$

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$$x^2 \sum_{n=0}^{\infty} a_n (x-1)^n = \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} a_n (x-1)^n$$

and so

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} a_n (x-1)^n &= \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} a_n (x-1)^n + 2(x-1) \sum_{n=0}^{\infty} a_n (x-1)^n \\ &\quad + (x-1)^2 \sum_{n=0}^{\infty} a_n (x-1)^n \end{aligned}$$

and so

$$
\begin{aligned}
x^2 \sum_{n=0}^{\infty} a_n (x-1)^n &= \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} a_n (x-1)^n + 2(x-1) \sum_{n=0}^{\infty} a_n (x-1)^n \\
&\quad + (x-1)^2 \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^{n+1} \\
&\quad + \sum_{n=0}^{\infty} a_n (x-1)^{n+2}
\end{aligned}
$$

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$$= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+2}$$

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$$\begin{aligned} &= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+2} \\ &= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} 2a_{n-1} (x-1)^n + \sum_{n=2}^{\infty} a_{n-2} (x-1)^n \end{aligned}$$

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\begin{aligned}
&= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} 2a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+2} \\
&= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} 2a_{n-1} (x-1)^n + \sum_{n=2}^{\infty} a_{n-2} (x-1)^n \\
&= a_0 + a_1 (x-1) + \sum_{n=2}^{\infty} a_n (x-1)^n \\
&\quad +2a_0 (x-1) \sum_{n=2}^{\infty} a_{n-1} (x-1)^n \\
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&= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} 2a_{n-1} (x-1)^n + \sum_{n=2}^{\infty} a_{n-2} (x-1)^n \\
&= a_0 + a_1 (x-1) + \sum_{n=2}^{\infty} a_n (x-1)^n \\
&\quad +2a_0 (x-1) \sum_{n=2}^{\infty} a_{n-1} (x-1)^n \\
&\quad + \sum_{n=2}^{\infty} a_n (x-1)^n \\
&= a_0 + (a_1 + 2a_0)(x-1) + \sum_{n=2}^{\infty} [a_n + 2a_{n-1} + a_{n-2}] (x-1)^n
\end{aligned}
$$

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$$

For this $y(x)$

$$
\begin{aligned}
y(1) &= 1 \quad \Rightarrow \quad a_0 = 1 \\
y'(1) &= 2 \quad \Rightarrow \quad a_1 = 2
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Example, Cont'd

Next, we compute power series expressions in standard form representing each term in the differential equation:

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$$\begin{aligned} y'' &= \sum_{n=0}^{\infty} n\,(n-1)\,a_n\,(x-1)^{n-2} \\ &= 0 + 0 + \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,(x-1)^n \end{aligned}$$

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Next, we compute power series expressions in standard form representing each term in the differential equation:

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\begin{aligned}
y'' &= \sum_{n=0}^{\infty} n(n-1)\,a_n\,(x-1)^{n-2} \\
&= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\,(x-1)^{n}
\end{aligned}
$$

$$
\begin{aligned}
-xy' &= (-1-(x-1)) \sum_{n=0}^{\infty} n a_n\,(x-1)^{n-1} \\
&= \sum_{n=0}^{\infty} -n a_n\,(x-1)^{n-1} + \sum_{n=0}^{\infty} -n a_n\,(x-1)^{n} \\
&= 0 + \sum_{n=0}^{\infty} -(n+1)\,a_{n+1}\,(x-1)^{n} + + \sum_{n=0}^{\infty} -n a_n\,(x-1)^{n}
\end{aligned}
$$

Example, Cont'd

$$qq - xy' = \sum_{n=0}^{\infty} \left[-(n+1)\, a_{n+1} - n a_n \right] (x-1)^n$$

Example, Cont'd

$$qq - xy' = \sum_{n=0}^{\infty} \left[-\left(n+1\right) a_{n+1} - n a_n \right] \left(x-1\right)^n$$

and, of course,

$$-y = \sum_{n=0}^{\infty} -a_n \left(x-1\right)^n$$

Example, Cont'd

Thus,

$$0 = y'' - xy' - y$$

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\begin{aligned}
0 &= y'' - xy' - y \\
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&\quad + \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1)\, a_{n+2} - (n+1)\, a_{n+1} - n a_n + a_n] (x-1)^n
\end{aligned}
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&\quad + \sum_{n=0}^{\infty} \left[-(n+1)\,a_{n+1} - na_n\right](x-1)^n \\
&\quad + \sum_{n=0}^{\infty} a_n\,(x-1)^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\,a_{n+2} - (n+1)\,a_{n+1} - na_n + a_n\right](x-1)^n
\end{aligned}
$$

and so

$$
a_{n+2} = \frac{(n+1)\,a_{n+1} + na_n - a_n}{(n+2)(n+1)} = \frac{(n+1)\,a_{n+1} + (n-1)\,a_n}{(n+2)(n+1)}
$$

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a_3 &= a_{1+2} = \frac{(2)\,a_2 + (0)\,a_1}{(2)\,(2)} = \frac{a_2}{2} = \frac{1}{4}
\end{aligned}
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and so

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y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots
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and so

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\begin{aligned}
y\,(x) &= a_0 + a_1\,(x-1) + a_2\,(x-1)^2 + a_3\,(x-1)^3 + \cdots \\
&= 1 + 2\,(x-1) + \frac{1}{2}\,(x-1)^2 + \frac{1}{4}\,(x-1)^3 + \cdots
\end{aligned}
$$