Math 2233 - Lecture 20

Agenda:

1. Announcement: Homework 9 is a Canvas Assignment (rather than a MyLab Math assignment)
2. Solutions via Power Series
3. Example
4. Convergence of Power Series
5. Singular Points
6. A Simple Criterion for Convergence of Power Series Solutions

Summary: The Power Series Method

Goal: Find a solution of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (*)$$

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1. If initial conditions are given, choose the expansion point x_0 to coincide with the value of x where the initial conditions are defined. E.g.,

$$\left.\begin{aligned} y(2) &= 1 \\ y'(2) &= 3 \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} x_0 &= 2 \\ a_0 &= 1 \\ a_1 &= 3 \end{aligned}\right.$$

2. Express each term of the differential equation as a power series in standard form
   - Differentiate power series term-by-term
   - The functions $p(x)$ and $q(x)$ must be replaced by their Taylor expansions about x_0 before multiplying power series.
   - Use shifts of summation indices to put power series expressions back in standard form.
   - Sometimes you have to write the initial terms of a power series separately from the infinite summation.
3. Power series in standard form are readily added together, so you can combine the power series expressions calculated in Step 2 to see that the differential equation implies a power series equation of the form

$$0 = \sum_{n=0}^{\infty} A_n \left(n, a_{n+2}, \ldots, a_0\right) \left(x - x_0\right)^n = 0$$

   which will in turn imply an infinite set of equations $A_n \left(n, a_{n+2}, \ldots\right) = 0$

4. Solve these equations for the highest coefficient that appears in them, say its a_{n+2}:

   $a_{n+2} =$ some function of n and the lower coefficients $a_{n-1}, \ldots, a_0$

5. The resulting equations will be naturally organized so that you can systematically compute the higher coefficients $a_2, a_3, \ldots$ in terms of the first two a_0 and a_1. Set $a_0 = c_1$ and $a_1 = c_2$ and then compute as many a_n as you need.
6. Collect together the terms with c_1 as a factor as $c_1 y_1(x)$ and those with c_2 as a factor as $c_2 y_2(x)$/ Then you'll be able to express the general solution of the ODE in the usual form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Example 1

Find the general solution of

$$xy'' - y = 0$$

and then find the solution satisfying

$$\begin{aligned} y(1) &= 1 \\ y'(1) &= 2 \end{aligned}$$

Since we are to eventually impose boundary conditions at $x = 1$, we shall look for power series solutions about $x = 1$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

Example 1, Cont'd

We have

$$\begin{aligned}
xy'' &= (1+(x-1))\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-1} \\
&= \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^{n} + \sum_{n=-1}^{\infty} (n+1)(n)a_{n+1}(x-1)^{n} \\
&= 0+0+\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^{n} \\
&\quad +0+\sum_{n=0}^{\infty} (n+1)(n)a_{n+1}(x-1)^{n} \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2}+n(n+1)a_{n+1}](x-1)^{n}
\end{aligned}$$

Example 1, Cont'd

Having obtained a power series expresssion **in standard form** for xy'' we can now combine it with the $-y$ term in the differential equation:

$$\begin{aligned} 0 &= xy'' - y \\ &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + n(n+1)\, a_{n+1} \right] (x-1)^n \\ &\quad + \sum_{n=0}^{\infty} (-1)\, a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + n(n+1)\, a_{n+1} - a_n \right] (x-1)^n \end{aligned}$$

Since a power series $\sum_{n=0}^{\infty} A_n (x-1)^n$ can equal 0 only when all of its coefficients A_n equal 0, we must have

$$(n+2)(n+1)\, a_{n+2} + n(n+1)\, a_{n+1} - a_n = 0 \quad , \quad n = 0, 1, 2, \ldots$$

Example 1, Cont'd

or

$$a_{n+2} = \frac{a_n - n(n+1)\,a_{n+1}}{(n+2)(n+1)} \qquad (RR[n])$$

These are the Recursion Relations for the problem.

To get the general solution, we do not assume any initital conditions that determine a_0 and a_1. Instead we set

$$\begin{aligned} a_0 &= c_1 \\ a_1 &= c_2 \end{aligned}$$

where c_1, c_2 are arbitrary constants.
We can now employ the recursion relations $RR[n]$ to determine the remaining coefficients.

Example 1, Cont'd

Let's start applying the Recursion Relations

$$a_{n+2} = \frac{a_n - n\left(n+1\right) a_{n+1}}{\left(n+2\right)\left(n+1\right)} \qquad (RR[n])$$

$$\begin{aligned}
& & a_0 &= c_1 \\
& & a_1 &= c_2 \\
RR\left[0\right] &\Rightarrow & a_2 &= a_{0+2} = \frac{a_0 - \left(0\right)\left(0+1\right) a_{0+1}}{\left(0+2\right)\left(0+1\right)} = \frac{c_1}{2} \\
RR\left[1\right] &\Rightarrow & a_3 &= a_{1+2} = \frac{a_1 - \left(1\right)\left(1+1\right) a_{1+1}}{\left(1+2\right)\left(1+1\right)} = \frac{a_1}{6} - \frac{a_2}{3} = \frac{c_2}{6} - \frac{c_1}{6} \\
RR\left[2\right] &\Rightarrow & a_4 &= a_{2+2} = \frac{a_2 - \left(2\right)\left(2+1\right) a_{2+1}}{\left(2+2\right)\left(2+1\right)} = \frac{a_2}{12} - \frac{a_3}{2} \\
& & &= \frac{c_1}{24} - \frac{1}{2}\left(\frac{c_2}{6} - \frac{c_1}{6}\right) = \frac{1}{8}c_1 - \frac{1}{12}c_2
\end{aligned}$$

Example 1, Cont'd

We'll now begin to write down the general solution

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots \\ &= c_1 + c_2(x-1) + \frac{c_1}{6}(x-1)^2 + \left(\frac{c_2}{6} - \frac{c_1}{6}\right)(x-1)^3 \\ &\quad + \left(\frac{1}{8}c_1 - \frac{1}{12}c_2\right)(x-1)^4 + \cdots \\ &= c_1 y_1(x) + c_2 y_2(x) \end{aligned}$$

where

$$\begin{aligned} y_1(x) &= 1 + (x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{8}(x-1)^4 + \cdots \\ y_2(x) &= (x-1) + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots \end{aligned}$$

are two linearly independent solutions.

Example 1, Cont'd

Let's now impose the initial conditions given at the start of the problem:

$$\begin{aligned} 1 &= y(1) = a_0 = c_1 \\ 2 &= y'(1) = a_1 = c_2 \end{aligned}$$

and so the solution satisfying the initial conditions is

$$\begin{aligned} y(x) &= (1)\, y_1(x) + (2)\, y_2(x) \\ &= 1 + (x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{8}(x-1)^4 + \cdots \\ &\quad +2\left((x-1) + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots\right) \\ &= 1 + 2(x-1) + (x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \cdots \end{aligned}$$

Convergence of Power Series

It is now time to discuss an important technical question:

Question: Given a function of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for what values of x is this a legitimate function?

To see the issue here, consider

$$y(x) = \sum_{n=0}^{\infty} x^n \qquad (a_n = 1 \text{ for all } n)$$

If we substitute $x = 1$ into this function we get

$$y(1) = \sum_{n=0}^{\infty} (1)^n \equiv \lim_{N\to\infty} \sum_{n=0}^{N} 1 = \lim_{N\to\infty} N = \infty$$

and so it does not make sense to evaluate this function at $x = 1$.

Convergent Power Series

Thus, when when we find a power series solution to a differential equation

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (*)$$

we need to make sure that

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x - x_0)^n \qquad (**)$$

actually exists before using (*) as a solution.

Definition

A power series (*) for which the limit on the right hand side of (**) exists for all x in a neighborhood of x_0 is called a **convergent power series**.

Facts about Convergent Power Series

Theorem

(i) *Suppose the limit on the right hand side of*

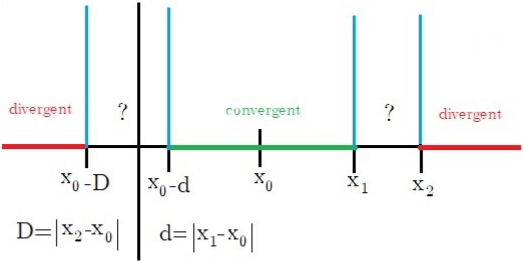
$$f(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n (x - x_0) \qquad (**)$$

exists when $x = x_1$*. Then the limit continues to exist for any* x *such that*

$$|x - x_0| < |x_1 - x_0|$$

(ii) *Conversely, suppose the limit on the right hand side of (**) does not exist for* $x = x_2$*. Then the limit also fails to exist for any* x *such that*

$$|x - x_0| > |x_1 - x_0|$$



Definition

The radius of convergence of a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (*)$$

is distance from the expansion point x_0 at which the power series transitions from a convergent power series to a divergent power series.

In other words, if R is the radius of convergence of

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (*)$$

then (*) defines a legitimate function of all x such that $|x - x_0| < R$. Thus,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (*)$$

does not makes sense as a function unless $x \in (x_0 - R, x_0 + R)$.

What I'll describe next is a simple way of figuring out the radius of convergence of a power series solution to

$$y ='' + p(x)y' + q(x)y = 0$$

Singular Points and the Convergence of Series Solutions

As it stands our method of finding power series solutions to differential equations of the form

$$y'' + p(x)y' + q(x)y = 0$$

is purely formal. For a series solution

$$\sum_{n=0}^{\infty} a_n (x - x_o)^n$$

might not converge for any x (and we need the series to converge if we are to use it to define a legitimate function of x).
To discuss this situation with the care it deserves, we must first introduce a little more formal development.

Analytic Functions

Definition

A function f is said to be **analytic** about the point x_o if $f(x)$ can be expressed as a convergent power series (e.g. by computing its Taylor expansion) near that point; i.e.,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

with some non-zero radius of convergence.

Theorem

If the functions $p(x)$ and $q(x)$ are analytic at the point x_o, then one can find (linear) functions a_n of a_o and a_1 so that the general solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

can be expressed as a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n(a_o, a_1)(x - x_o)^n = a_o y_1(x) + a_1 y_2(x) \quad ,$$

where y_1 and y_2 are two linearly independent solutions of (1) which are analytic at x_o. Moreover, the radius of convergence of the power series expansions of y_1 and y_2 is at least as large as the minimum of the radii of convergence of the power series expressions (about x_0) for $p(x)$ and $q(x)$

Thus, if we know the radii of convergence $p(x)$ and $q(x)$ we needn't do anything as laborious as compute the radius of convergence of our solution using things like the ratio test. We just need to figure out the radii of convergence of the power series expansions of $p(x)$ and $q(x)$ about x_0.

Theorem

If $f(x)$ is the ratio of two polynomial functions;

$$f(x) = \frac{P(x)}{Q(x)}$$

and $Q(x_o) \neq 0$, then

(i) *$f(x)$ has a power series expansion about $x = x_o$.*

(ii) *The radius of convergence of this power series about x_o is equal to the distance* **in the complex plane** *between x_o and the nearest zero of $Q(x)$.*

Example

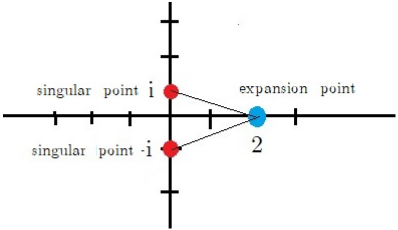
What is the radius of convergence of the Taylor expansion of

$$f(x) = \frac{1}{1 + x^2}$$

about $x = 2$?

The denominator vanishes when $x = \pm i$. To determine the radius of convergence we need only compute the distance in the complex plane between $x = \pm i$ and the expansion point.

Here is a picture of the situation:



In terms of the Cartesian coordinates of the complex plane ($z \in \mathbb{C} \to z = x + iy$)) the points $z = \pm i$ are given by, respectively, (0,1) and (0,-1), while the coordinates of the real number $z = 2 = 2 + (0)i$ are given by (2,0).
The distance between 2 and $\pm i$ is then

$$\sqrt{(2-0)^2 + (0 \mp 1)^2} = \sqrt{5} \quad ,$$

so the radius of convergence of the Taylor series expansion of $\frac{1}{1+x^2}$ about $x = 2$ is $\sqrt{5}$.

Example

Find the radius of convergence of the Taylor series expansion of

$$f(x) = \frac{1}{(x+2)(x-3)} \qquad \text{(I26-05)}$$

about $x_o = 4$.
The zeros of the denominator are $x = -2, 3$. The distance (in the complex plane from $x_o = 4 = (4, 0)$ to the closest zero $x = 3 = (3, 0)$ is

$$\sqrt{(4-3)^2 - (0-0)^2} = 1 \quad ,$$

so the radius convergence of the Taylor series expansion of $f(x)$ about $x_o = 4$ is 1.

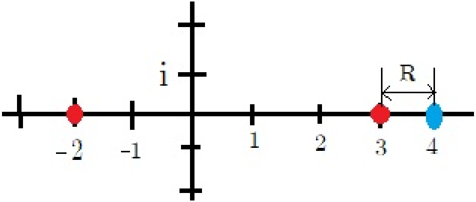
Let us now combine the two theorems to determine the minimal radius of convergence of the power series solution of

$$(x^2 - 2x - 3)y'' + xy' + 4(x - 3)y = 0 \qquad \text{(I26-06)}$$

about $x_o = 4$.
This differential equation is equivalent to

$$y'' + \frac{x}{x^2 - 2x - 3}y' + \frac{4}{x + 2}y = 0 \quad .$$

The zeros of $x^2 - 2x - 3 = (x - 3)(x + 2)$ are $x = 3, -2$, and -2 is the only zero $x + 2$. So the singular points are $x = -2$ and $x = 3$. Below is a picture of the situation:



The singular point that's closest to the expansion point $x = 4$ is obviously the one at $x = 3$.
Since $|4 - 3| = 1$, the radius of convergence of a power series solution about $x_o = 4$ is 1.
Thus, if

$$y(x) = \sum_{n=0}^{\infty} a_n (x - 4)^n$$

is a power series solution of

$$\left(x^2 - 2x - 3\right) y'' + xy' + 4(x - 3) y = 0$$

then $y(x)$ will be a well-defined function of x only when $3 < x < 5$.

Summary: Determining the Radius of Convergence of Power Series Solutions

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is a power series solution of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

where

$$p(x) = \frac{A(x)}{B(x)} \quad , \quad q(x) = \frac{C(x)}{D(x)}$$

1. Determine the zeros of the denominators $B(x)$ and $D(x)$ in the complex plane $\mathbb{C}$. Let's say they are $\{z_1, \ldots, z_k\}$
2. Pick the z_j that's closest to x_0 in the complex plane (using Cartesian coordinates (x, y) for $z_j = x_j + iy_j$ to compute distances in $\mathbb{C}$).
3. $R = \|z_j - x_0\| = \sqrt{(x_j - x_0)^2 + (y_j - 0)^2}$