Math 2233 - Lecture 20

Agenda:

1. Announcement: Homework 9 is a Canvas Assignment (rather than a MyLab Math assignment)
2. Solutions via Power Series
3. Example
4. Convergence of Power Series
5. Singular Points
6. A Simple Criterion for Convergence of Power Series Solutions

Summary: The Power Series Method

Goal: Find a solution of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (*)$$

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which will in turn imply an infinite set of equations $A_n \left(n, a_{n+2}, \ldots\right) = 0$

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5. The resulting equations will be naturally organized so that you can systematically compute the higher coefficients $a_2, a_3, \ldots$ in terms of the first two; a_0 and a_1. Set $a_0 = c_1$ and $a_1 = c_2$ and then compute as many a_n as you need.

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6. Collect together the terms with c_1 as a factor as $c_1 y_1(x)$ and those with c_2 as a factor as $c_2 y_2(x)$/ Then you'll be able to express the general solution of the ODE in the usual form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Example 1

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Since we are to eventually impose initial conditions at $x = 1$, we shall look for power series solutions about $x = 1$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

Example 1, Cont'd

We have

$$xy'' = (1 + (x-1)) \sum_{n=0}^{\infty} n\,(n-1)\,a_n\,(x-1)^{n-2}$$

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&= \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^{n} + \sum_{n=-1}^{\infty} (n+1)(n)a_{n+1}(x-1)^{n}
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&= \sum_{n=0}^{\infty} \left[(n+2)(n+1)\,a_{n+2} + n(n+1)\,a_{n+1}\right](x-1)^{n}
\end{aligned}
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0 &= xy'' - y \\
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&= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n(n+1) a_{n+1} - a_n \right] (x-1)^n
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Since a power series $\sum_{n=0}^{\infty} A_n (x-1)^n$ can equal 0 only when all of its coefficients A_n equal 0, we must have

$$(n+2)(n+1) a_{n+2} + n(n+1) a_{n+1} - a_n = 0 \quad , \quad n = 0, 1, 2, \ldots$$

Example 1, Cont'd

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$$a_{n+2} = \frac{a_n - n\,(n+1)\,a_{n+1}}{(n+2)\,(n+1)} \qquad (RR[n])$$

These are the Recursion Relations for the problem.

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where c_1, c_2 are arbitrary constants.
We can now employ the recursion relations $RR[n]$ to determine the remaining coefficients.

Example 1, Cont'd

Let's start applying the Recursion Relations

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$$a_{n+2} = \frac{a_n - n\left(n+1\right)a_{n+1}}{\left(n+2\right)\left(n+1\right)} \qquad (RR[n])$$

$$\begin{aligned} & & a_0 &= c_1 \\ & & a_1 &= c_2 \\ RR\left[0\right] & \Rightarrow & a_2 &= a_{0+2} = \frac{a_0 - \left(0\right)\left(0+1\right)a_{0+1}}{\left(0+2\right)\left(0+1\right)} = \frac{c_1}{2} \end{aligned}$$

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RR[0] & \Rightarrow & a_2 &= a_{0+2} = \frac{a_0 - (0)(0+1)\,a_{0+1}}{(0+2)(0+1)} = \frac{c_1}{2} \\
RR[1] & \Rightarrow & a_3 &= a_{1+2} = \frac{a_1 - (1)(1+1)\,a_{1+1}}{(1+2)(1+1)} = \frac{a_1}{6} - \frac{a_2}{3} = \frac{c_2}{6} - \frac{c_1}{6}
\end{aligned}$$

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RR\,[2] & \Rightarrow & a_4 &= a_{2+2} = \frac{a_2 - (2)\,(2+1)\,a_{2+1}}{(2+2)\,(2+1)} = \frac{a_2}{12} - \frac{a_3}{2}
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& & &= \frac{c_1}{24} - \frac{1}{2}\left(\frac{c_2}{6} - \frac{c_1}{6}\right) = \frac{1}{8}c_1 - \frac{1}{12}c_2
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where

$$\begin{aligned} y_1(x) &= 1 + (x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{8}(x-1)^4 + \cdots \\ y_2(x) &= (x-1) + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots \end{aligned}$$

are two linearly independent solutions.

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Let's now impose the initial conditions given at the start of the problem:

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and so it does not make sense to evaluate this function at $x = 1$.

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Thus, when when we find a power series solution to a differential equation

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we need to make sure that

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0)^n \tag{**}$$

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$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (*)$$

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Definition

A power series (*) for which the limit on the right hand side of (**) exists for all x in a neighborhood of x_0 is called a **convergent power series**.

Facts about Convergent Power Series

Theorem

(i) *Suppose the limit on the right hand side of*

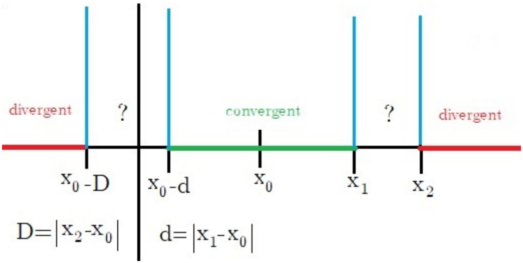
$$f(x) = \lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0) \qquad (**)$$

exists when $x = x_1$. Then the limit continues to exist for any x such that

$$|x - x_0| < |x_1 - x_0|$$

(ii) *Conversely, suppose the limit on the right hand side of (**) does not exist for $x = x_2$. Then the limit also fails to exist for any x such that*

$$|x - x_0| > |x_2 - x_0|$$



Definition

The radius of convergence of a power series

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n \qquad (*)$$

is the distance R from the expansion point x_0 at which the power series transitions from a convergent power series to a divergent power series.

In other words, if R is the radius of convergence of

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What I'll describe next is a simple way of figuring out the radius of convergence of a power series solution to

$$y'' + p(x)y' + q(x)y = 0$$

Singular Points and the Convergence of Series Solutions

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To discuss this situation with the care it deserves, we must first introduce a little more formal development.

Analytic Functions

Definition

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A function f is said to be **analytic** about the point x_o if $f(x)$ can be expressed as a convergent power series (e.g. by computing its Taylor expansion) near that point; i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_o)^n$$

with some non-zero radius of convergence.

Theorem

If the functions $p(x)$ and $q(x)$ are analytic at the point x_o, then one can find (linear) functions a_n of a_o and a_1 so that the general solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

can be expressed as a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n(a_o, a_1)(x - x_o)^n = a_o y_1(x) + a_1 y_2(x) \quad ,$$

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where y_1 and y_2 are two linearly independent solutions of (1) which are analytic at x_o. Moreover, the radius of convergence of the power series expansions of y_1 and y_2 is at least as large as the minimum of the radii of convergence of the power series expressions (about x_0) for $p(x)$ and $q(x)$

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(i) *$f(x)$ has a power series expansion about $x = x_o$.*

(ii) *The radius of convergence of this power series about x_o is equal to the distance* **in the complex plane** *between x_o and the nearest zero of $Q(x)$.*

Example

What is the radius of convergence of the Taylor expansion of

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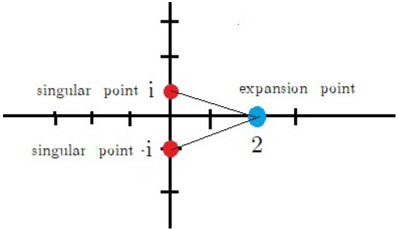
The denominator vanishes when $x = \pm i$. To determine the radius of convergence we need only compute the distance in the complex plane between $x = \pm i$ and the expansion point.

Here is a picture of the situation:



In terms of the Cartesian coordinates of the complex plane $(z \in \mathbb{C} \to z = x + iy))$ the points $z = \pm i$ are given by, respectively, (0,1) and (0,-1), while the coordinates of the real number $z = 2 = 2 + (0)i$ are given by (2,0).

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so the radius of convergence of the Taylor series expansion of $\frac{1}{1+x^2}$ about $x = 2$ is $\sqrt{5}$.

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so the radius convergence of the Taylor series expansion of $f(x)$ about $x_o = 4$ is 1.

Let us now combine the two theorems to determine the minimal radius of convergence of the power series solution of

$$(x^2 - 2x - 3)y'' + xy' + 4(x - 3)y = 0 \qquad \text{(I26-06)}$$

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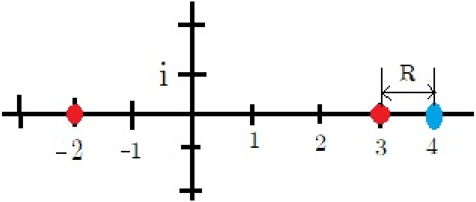

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then $y(x)$ will be a well-defined function of x only when $3 < x < 5$.

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3. $R = \|z_j - x_0\| = \sqrt{(x_j - x_0)^2 + (y_j - 0)^2}$