Agenda:

1. Convergence of Power Series Solutions : Reprise
2. Singular Points of Differential Equations
3. Regular vs. Irregular Singular Points
4. Generalized Power Series Solutions

Convergence of Power Series Solutions

In Lecture 20, it was pointed out that a power series function, in particular, a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{1}$$

of a differential equation

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

may or may not define a legitimate function.

The issue here is that for some values of x it may happen that

$$y(x) \equiv \lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0)^n \tag{3}$$

may not exist.

However, it turns out that the range of x for which the limit in (3) exists, can be determined from the coefficients functions $p(x)$ and $q(x)$ in the differential equation (2)

Suppose

$$p(x) = \frac{A(x)}{B(x)} \quad , \quad q(x) = \frac{C(x)}{D(x)}$$

are rational functions (so $A(x), B(x), C(x), D(x)$ are all polynomials in x).

Then we have the following procedure for determining the range of x for which a solution (1) of (2) is valid.

1. Find the zeros of the denominators $B(x)$ and $D(x)$ in the complex plane. Let's label them $\{z_1, \ldots, z_k\}$. We'll refer to these points as the **singular points** of the differential equation.
2. Calculate the distance between the each points z_i and the expansion point x_0 of your solution (3).
3. The shortest of the distances calculate in Step 2 will be the (minimal) radius of convergence R of a solution of the form (1).
4.
5. And so a solution of the form (3) will be a valid function for all $x \in (x_0 - R, x_0 + R)$.

Example

Consider the ODE

$$\left(x^2-2x-3\right)y''+xy'+(x+1)\,y=0$$

Find the interval $(a,b)\subset\mathbb{R}$ for which a power series solution of the form

$$y(x)=\sum_{n=0}^{\infty}a_n\,(x-2)^n$$

is defined.

The ODE in standard form we have

$$y''+\frac{x}{x^2-2x-3}y'+\frac{x+1}{x^2-2x-3}y=0$$

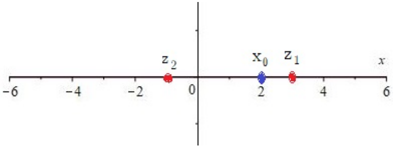
and so

$$\begin{aligned}
p(x) &= \frac{x}{x^2-2x-3}=\frac{x}{(x-3)(x+1)}\\
q(x) &= \frac{x+1}{x^2-2x-3}=\frac{x+1}{(x-3)(x+1)}=\frac{1}{x-3}
\end{aligned}$$

We see that $p(x)$ is undefined when $x = 3$ and when $x = -1$ (because at those points the denominator of $p(x)$ is 0), and $q(x)$ is undefined at $x = 3$.

The singular points of the differential equation are thus $z_1 = 3$ and $z_2 = -1$.

Below is a picture of the situation:



Clearly, the singular point closest to the expansion point $x_0 = 2$ is $z_1 = 3$. Hence, the radius of convergence of a power series solution about $x = 2$ will be

$$R = |2 - 3| = 1$$

Conclusion: A series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - 2)^n$$

will therefore be valid on the interval $(2 - R, 2 + R) = (1, 3)$.

More on Singular Points

Definition

A (in general, complex) number z is called a **singular point** of the differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{4}$$

if either

$$\lim_{x \to z} p(x) \text{ does not exist, or}$$
$$\lim_{x \to z} q(x) \text{ does not exist}$$

From the preceding discussion, we know that power series solutions of ODEs such as (4) inevitably fail to make sense as functions of x as we approach a singular point.

This does not necessarily mean, however, that we don't have valid solutions near singular points.

Example

$$x^2y'' - xy' - 3y = 0$$

This is an Euler-type equation, and so we can expect solutions of the form $y(x) = x^r$. Its auxiliary equation is

$$0 = r^2 + (-1 - 1)\,y - 3 = r^2 - 2r - 3 = (r - 3)(r + 1)$$

and so we have the following two independent solutions

$$\begin{aligned} y_1(x) &= x^3 \\ y_2(x) &= x^{-1} \end{aligned}$$

OTOH, the differential equation in standard form is

$$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = 0$$

and so has it has a singular point at $x = 0$. Note that one solution $y_1(x) = x^3$ makes perfectly good sense near the singular point at $x = 0$, while the other solution $y_2(x) = \frac{1}{x}$ is undefined at $x = 0$.

Moral of the Example

Thus,

- Just because a differential equation has a singular point at x_1, doesn't mean that it doesn't have a valid solution near x_1.
- On the other hand, having a singular point at x_1, can still cause problems for solutions near x_1.

What I aim to show you next is that so long as the singularities of the coefficient functions $p(x)$ and $q(x)$ are not too bad (in a sense that will be made precise in a second), we will be able extend our power series technique to find **generalized power series solutions** that are valid functions right up to (but sometimes not including) the singular point.

Regular Singular Points

Here's what I mean by "a singularity not being too bad".

Definition

Consider a differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

with a singular point at $x = z$. If

- the singularity of $p(x)$ as x approaches z is no worse than that of $\frac{1}{x-z}$, and
- the singularity of $q(x)$ as x approaches z is no worse than that of $\frac{1}{(x-z)^2}$

Then z is said to be a **regular singular point**. Otherwise, it is said to be an **irregular singular point**.

Alternative Definitions

We say that a function $f(x)$ has a **singular point** at $x = z$ if

$$\lim_{x \to z} f(x) \quad \text{does not exist}$$

The **degree** of a singularity of $f(x)$ at z is the smallest number n such that

$$\lim_{x \to z} (x - z)^n f(x) \quad \text{does exist}$$

Note that the factor $(x - z)^n$ in $(x - z)^n f(x)$ can be used to cancel as many as n factors of $\frac{1}{x-z}$ in the denominator of $f(x)$. So if $f(x)$ is of the form

$$f(x) = \frac{g(x)}{(x - z)^n}$$

with $g(z) \neq 0$, then

$$deg\,(f, z) = n$$

Using this new nomenclature, z will a regular singular point of

$$y'' + p(x)y' + q(x)y = 0$$

if either $deg(p(x), z) > 0$ or $deg(q(x), z) > 0$ (so z is a singular point of the ODE) and we have both

$$deg\,(p(x), z) \leq 1$$
$$deg\,(q(x), z) \leq 2$$

Here is the reason for focusing on regular singular points.

Theorem

If x_1 is a regular singular point of a differential equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

then the differential equation will have at least one solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_1)^{n+r}$$

More over, such a generalized power series solution will be valid at all points sufficiently close to x_1; except perhaps at $x = x_1$ itself.

I'll show you how generalized power series work in a minute. But first let's get some practice in identifying regular singular points.

Example 1

The differential equation

$$y'' + \frac{3}{(x-1)(x+1)^2}y' + \frac{2x+1}{(x-2)^2(x+2)(x-1)^3}y = 0$$

has singular points at $x = 1, -1, 2, -2$. Now

z	$deg\,(p(x), z) \leq^? 1$	$deg\,(q(x), z) \leq^? 2$	Type
1	$1 \leq 1$ ✓	$3 \not\leq 2$!	irregular
-1	$2 \not\leq 1$!	$0 \leq 2$ ✓	irregular
2	$0 \leq 1$ ✓	$2 \leq 2$ ✓	regular
-2	$0 \leq 1$ ✓	$1 \leq 2$ ✓	regular

So $x = \pm 1$ are irregular singular points and $x = \pm 2$ are regular singular points.

Example 2

Identify and classify the singular points of

$$x^2(1-x^2)^2y'' + x(1+x)^2y' + (1-x)y' \quad . \tag{1}$$

In this case, when we divide by $x^2(1-x^2)^2$ to put the equation in standard form, we have

$$p(x) = \frac{x(1+x)(1+x)}{x^2(1+x)^2(1-x)^2} = \frac{1}{x(1-x)^2} \tag{2}$$

and

$$q(x) = \frac{(1-x)}{x^2(1+x)^2(1-x)^2} = \frac{1}{x^2(1+x)^2(1-x)} \quad . \tag{3}$$

Thus, we have regular singular points at $x = 0, -1$ and an irregular singular point at $x = 1$.

Solutions of Bessel's Equation

Bessel's equation in a 2nd order linear ODE that often arises when solving systems coordinatized by spherical coordinates.
Bessel's equation is actually a family of diffential equations

$$R'' + \frac{1}{r}R' + \left(n^2 - \frac{m^2}{r^2}\right)R = 0 \tag{5}$$

where n and m are integers. Note that it has a regular singular point at $r = 0$.
So that the main ideas of the generalized power series technique are presented as simply as possible, we'll focus on the special case where $n = 1$ and $m = 0$;

$$r^2 R'' + rR' + r^2 R = 0 \tag{6}$$

Solutions of (6) are called *Bessel functions of order 0*.

Let us rewrite (6) as

$$x^2y'' + xy' + x^2y = 0 \tag{3}$$

(just changing the labels of variables to the way we usually write an ODE in this course). We shall look for solutions of (3) of the form

$$\begin{aligned} y &= x^r \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots \\ &= \sum_{n=0}^{\infty} a_n x^{n+r} \end{aligned}$$

We can assume (without loss of generality) that $a_0 \neq 0$; so that $a_0 x^r$ really is the leading term of this solution.

We have

$$
\begin{aligned}
x^2 y'' &= x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \\
&= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} \\
x y' &= x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} \\
x^2 y &= x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
\end{aligned}
$$

And so when we replace x^2y'', xy' and x^2y with their series expressions we get

$$0=\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

The first two series begin two steps before the last series, so before we can combine the power series we have to "peel off" the two initial terms of the first two series:

$$\begin{aligned}
0 &= (r)(r-1)\,a_0x^r+(r+1)(r)\,a_1x^{r+1} \\
&\quad +\sum_{n=2}^{\infty}(n+r)(n+r-1)\,a_nx^{n+r} \\
&\quad +ra_0x^r+(r+1)\,a_1x^{r+1}+\sum_{n=2}^{\infty}(n+r)\,a_nx^{n+r} \\
&\quad +\sum_{n=2}^{\infty}a_{n-2}x^{n+r} \\
&= [r(r-1)+r]\,a_0x^r+(r(r+1)+r+1)\,a_1x^{r+1} \\
&\quad +\sum_{n=2}^{\infty}[(n+r)(n+r-1)\,a_n+(n+r)\,a_n+a_{n-2}]\,x^{n+r}
\end{aligned}$$

Note that we have now stratified the right hand side as sum of terms, each term having a distinct power of x as a factor.
We now demand that the total coefficient of each power of x separately vanish.
The lowest order term is

$$[r(r-1)+r]\,a_0x^r = r^2a_0x^r$$

For this to vanish for all x we need

$$r^2a_0 = 0 \quad \Rightarrow \quad r = 0$$

since our ansatz for y assumes that $a_0 \neq 0$.
The next higher order term is (using $r = 0$)

$$(r(r+1)+r+1)\,a_1x^{r+1} = (0(0+1)+0+1)\,a_1x = a_1x$$

So if this to vanish for all x we must have $a_1 = 0$.

Let's now look at total coefficient of x^{n+r} in the sum $\sum_{n=2}^{\infty}$. This must vanish, and so

$$\begin{aligned} 0 &= \left[(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2}\right] \\ &= \left[n(n-1)a_n + na_n + a_{n-2}\right] \\ &= na_n + a_{n-2} \end{aligned}$$

and so

$$a_n = \frac{a_{n-2}}{n} \quad , n = 2, 3, 4, \ldots$$

These are our Recursion Relations.
And we have

$$\begin{aligned} r &= 0 \\ a_1 &= 0 \end{aligned}$$

coming from the two leading terms of our expression of the differential equation as a generalized power series equation.

Using the recursion relations

$$a_n = \frac{-a_{n-2}}{n} \quad , \quad n = 2, 3, 4, 5, \ldots$$

we can now begin to write down a solution. We have

$$\begin{aligned}
a_0 &= \text{arbitrary constant} \\
a_1 &= 0 \\
a_2 &= \frac{-a_0}{2} \\
a_3 &= \frac{-a_1}{3} = -\frac{0}{3} = 0 \\
a_4 &= \frac{-a_2}{4} = \frac{a_0}{4 \cdot 2} = \frac{a_0}{2^2 2!} \\
a_5 &= -\frac{a_3}{2} = 0 \\
a_6 &= -\frac{a_4}{6} = -\frac{a_0}{6 \cdot 4 \cdot 2} = -\frac{a_0}{2^3 3!}
\end{aligned}$$

We thus observe the following pattern

$$a_n = \begin{cases} (-1)^k \frac{a_0}{2^k k!} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

And so we can write

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k}$$

as the solution to (3).

Note that this solution continues to make sense even as we let x approach the regular singular point at $x = 0$.