Lecture 23 : Review and Summary

Agenda:

I. Differential Equations: Solutions and Classification

II. 1st Order Differential Equations - Approximate Methods

III. 1st Order Differential Equations - Exact Methods

IV. 2nd Order Linear Ordinary Differential Equations: General Theory

V. Laplace Transform Method

VI. Power Series Solutions of 2nd Order Linear ODEs

I. Differential Equations: Solutions and Classification

- Ordinary Differential Equations (ODEs) vs. Partial Differential Equations (PDEs)
- The Order of a Differential Equation
- Linear vs. Nonlinear ODEs

II. First Order Differential Equations - Approximate Methods

- General Form

$$\frac{dy}{dx} = F(x, y)$$

- Direction Fields and the Graphical Method
- Numerical Methods

$$\frac{dy}{dx} = F\left(x, y\right) \quad , \quad y\left(x_0\right) = y_0$$

$$\begin{aligned} x_0 &= x_0 \\ y_0 &= y_0 \end{aligned}$$

$$\begin{aligned} x_{i+1} &= x_i + \Delta x \\ y_{i+1} &= y_i + F\left(x_i, y_i\right) \Delta x \end{aligned}$$

III. First Order Differential Equations - Exact Methods

- General Solution vs. Unique Solution to Initial Value Problem
- Fundamental Theorem of Calculus

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad y(x) = \int f(x)\,dx + C$$

- Separable Equations

$$M(x) + N(y)\frac{dy}{dx} = 0 \quad \Rightarrow \quad \int M(x)\,dx + \int N(y)\,dy = C$$

- First Order Linear ODEs

$$y' + p(x)\,y = g(x) \quad \Rightarrow \quad y(x) = \frac{1}{\mu(x)}\int \mu(x)\,g(x)\,dx + \frac{C}{\mu(x)}$$

where $\mu(x)$ is the "integrating factor"

$$\mu(x) = \exp\left[\int p(x)\,dx\right]$$

Remember

$$\exp(\lambda \ln(x)) = x^{\lambda}$$

III. First Order Differential Equations - Exact Methods, Cont'd

- Exact Equations

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Phi(x,y) = \begin{cases} \int M(x,y)\,\partial x + h_1(y) \\ \int N(x,y)\,\partial y + h_2(x) \end{cases}$$

Figure out correct choice for arbitrary functions $h_1(y), h_2(x)$
Solve $\Phi(x,y) = C$ for $y(x)$

IV. 2nd Order Linear ODEs:

- Standard Forms; Homogeneous vs. Nonhomogeneous Cases

$$\begin{aligned} y'' + p(x)\,y' + q(x)\,y &= 0 && (0) \\ y'' + p(x)\,y' + q(x)\,y &= g(x) && (1) \end{aligned}$$

Homogeneous 2nd Order Linear ODEs:

$$y'' + p(x)y' + q(x)y = 0 \tag{0}$$

- Superposition Principle: If $y_1(x)$ and $y_2(x)$ are solutions of (0) , then so is $y(x) = c_1 y_1(x) + c_2 y_2(x)$
- Form of General Solution: Every solution of (0) is of the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where

$$0 \neq W[y_1, y_2](x) = y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)$$

- Reduction of Order Formula:

$$y_2 = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\, dx\right]$$

The Simple Cases of Homogeneous Linear ODEs

- Constant Coefficients Case: $ay'' + by' + cy = 0$.

$$
\begin{aligned}
y(x) &= e^{\lambda x} \\
\Rightarrow \quad & a\lambda^2 + b\lambda + c = 0 \\
y(x) &= \begin{cases} c_1 e^{\lambda_1 x} + c_2 e^{\lambda_1 x} & \lambda_1, \lambda_2 \in \mathbb{R} \\ c_1 e^{\lambda x} + c_2 x e^{\lambda x} & \lambda \in \mathbb{R} \\ c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) & \lambda = \alpha \pm i\beta \in \mathbb{C} \end{cases}
\end{aligned}
$$

- Euler-type Case: $ax^2 y'' + bxy' + cy = 0$.

$$
\begin{aligned}
y(x) &= x^m \\
\Rightarrow \quad & am^2 + (b-a)m + c = 0 \\
y(x) &= \begin{cases} c_1 x^{m_1} + c_2 x^{m_2} & m_1, m_2 \in \mathbb{R} \\ c_1 x^m + c_2 x^m \ln|x| & m \in \mathbb{R} \\ c_1 x^{\alpha} \cos(\beta \ln|x|) + c_2 x^{\alpha} \sin(\beta \ln|x|) & m = \alpha \pm i\beta \in \end{cases}
\end{aligned}
$$

Nonhomogeneous 2nd Order Linear ODEs:

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

- Form of the General Solution:

$$Y(x) = Y_p(x) + c_1 y_1(x) + c_2 y_2(x)$$

- Variation of Parameters Formula:

$$Y_p(x) = -y_1(x)\int \frac{y_2(x)\, g(x)}{W[y_1, y_2]}dx \quad + \quad y_2(x)\int \frac{y_1(x)\, g(x)}{W[y_1, y_2]}dx$$

V. Laplace Transform Method

- Laplace Transforms

$$\begin{aligned}
\mathcal{L}[f](s) &\equiv \int_0^\infty f(x)\, e^{-sx} dx \\
\mathcal{L}[y'] &= s\mathcal{L}[y] - y(0) \\
\mathcal{L}[y''] &= s^2\mathcal{L}[y] - sy(0) - y'(0)
\end{aligned}$$

- Inverse Laplace Transforms
  - Partial Fractions Expansions
  - Completing the Square in the Denominator
- Using Laplace Transform to Solve ODEs

VI. Power Series Solutions of 2nd Order Linear ODEs

- Trial solution: $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$
- Initial Conditions

$$\begin{aligned} a_0 &= y(x_0) \\ a_1 &= y'(x_0) \end{aligned}$$

- The DE determines $a_2, a_3, \ldots$ via its Recursion Relations

Power Series Manipulations

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \quad , \quad y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \\
q(x)\, y(x) &= \left(q(x_0) + q'(x_0)(x - x_0) + \frac{q''(x_0)}{2!}(x - x_0)^2 + \cdots \right) \\
&\quad * \sum_{n=0}^{\infty} a_n (x - x_0)^n
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} a_n (x - x_0)^n + \sum_{n=0}^{\infty} b_n (x - x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n
$$

$$
0 = \sum_{n=0}^{\infty} A_n (x - x_0)^n \quad \text{for all } x \quad \Rightarrow \quad A_n = 0 \text{ for all } n
$$

Power Series Method

- Choose expansion point x_0
- Substitute $y(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ into the ODE
- Manipulate the resulting equation to get it in the form $0 = \sum_{n=0}^{\infty} A_n(n, a_i) \left(x - x_0\right)^n$
- Use $A_n = 0$ to get the Recursion Relations
- Systematically solve the Recursion Relations to find $a_2, a_3, \ldots$

Singular Points and Convergence of Series Solutions

The radius of convergence of a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

to

$$y'' + p(x) y' + q(x) y = 0$$

will be the distance (in the complex plane) between x_0 and the closest singularity of $p(x)$ and $q(x)$