Math 3013
SOLUSTIONS TO FIRST EXAM
9:30 - 10:20, February 18, 2022

1. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Compute the matrix products **AB** and **BA** (if they exist).

- $$\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \text{ is undefined since } \#Columns\,(\mathbf{A}) \neq \#Rows\,(\mathbf{B})$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+0 \\ 2-1 & 0+1 \\ 4+1 & 0-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 5 & -1 \end{bmatrix}$$

2. For each of the following augmented matrices, indicate

(i) the number of equations and the number of variables in the original linear system
(ii) whether or not the corresponding linear system has a solution
(iii) if the corresponding linear system does have a solution, the number of free variables in the solution.

Hint: Note that these augmented matrices are already in row echelon form and that you do not have to do any calculations to answer the questions.

(a) $$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(i) 3 equations in 4 unknowns
(ii) no solution (last row implies $0 = 1$ which indicates there was a contradiction amongst the original equations)

(b) $$\left[\begin{array}{cccc|c} 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

(i) 3 equations in 4 unknowns
(ii) there is a solution
(iii) Column 1 of the REF lacks a pivot while the other columns on the left have pivots. This implies there is 1 free variable (x_1) in the solution.

(c) $$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(i) 4 equations in 5 unknowns
(ii) there is a solution
(iii) Columns 3 and 5 on the left lack pivots. This implies there will be two free variables (x_3 and x_5) in the general solution.

3. Consider the following linear system

$$\begin{aligned} -x_2 - 3x_3 &= -5 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Write down the corresponding augmented matrix and row reduce it to row-echelon form.

- $$\begin{aligned} [\mathbf{A} \mid \mathbf{b}] &= \left[\begin{array}{ccc|c} 0 & -1 & -3 & -5 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{array}\right] \\ &\xrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 1 & 1 & 1 \\ 0 & -1 & -3 & -5 \end{array}\right] \\ &\xrightarrow{R_2 \longleftrightarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -1 & -3 & -5 \\ 0 & -1 & -3 & -5 \end{array}\right] \\ &\xrightarrow{R_3 \longleftrightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -1 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad (R.E.F.) \end{aligned}$$

4. Row reduce the following augmented matrix to **reduced** row-echelon form.

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 6 & 2 & 2 \\ 0 & 0 & -1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

, row echelon form: $\left[\begin{array}{cccccc} 1 & 2 & 0 & 14 & 0 & 24 \\ 0 & 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- $$\begin{aligned} &\xrightarrow[R_3 \to -\frac{1}{2}R_3]{R_2 \to -R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 6 & 2 & 2 \\ 0 & 0 & 1 & -2 & -2 & -3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 6 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{R_1 \to R_1 - 4R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 14 & 0 & 24 \\ 0 & 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad (R.R.E.F.) \end{aligned}$$

5. Suppose the augmented matrix below is the Reduced Row Echelon Form of an augmented matrix of a linear system. Display the solution of the linear system as a hyperplane (within the space of variables).

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- There are no pivots in columns 2, 4 or 5. This means that x_2, x_4, and x_5 will be free variables in the solution. Writing down the equations corresponding to this augmented matrix in R.R.E.F., and then solving for the "fixed variables" x_1 and x_3 yields

$$\left.\begin{array}{r} x_1 + x_2 - 2x_4 + x_5 = 1 \\ x_3 - x_4 - x_5 = 2 \\ 0 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{r} x_1 = 1 - x_2 + 2x_4 - x_5 \\ x_3 = 2 + x_4 + x_5 \end{array}\right.$$

Next, we try writing down a typical solution vector, substituting where we can for the fixed variables, and then expanding the result in terms of the free variables yields:

$$\mathbf{x} = \left[\begin{array}{l} 1 - x_2 + 2x_4 - x_5 \\ x_2 \\ 2 + x_4 + x_5 \\ x_4 \\ x_5 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{array}\right] + x_2 \left[\begin{array}{c} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right] + x_4 \left[\begin{array}{c} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}\right] + x_5 \left[\begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}\right]$$

The last expression on the right exhibits the solution as a 3-dimensional hyperplane in $\mathbb{R}^5$.

6. Determine if the following matrix has an inverse. If the matrix does have an inverse, determine its inverse. (Hint: a single row reduction process can determine the answer to both parts of this problem.)

$$\mathbf{A} = \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{array}\right]$$

- We will form the adjoined matrix $[\mathbf{A} \mid \mathbf{I}]$ and then row reduce it to its Reduced Row Echelon Form:

$$\begin{aligned} [\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \\ &\xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \\ &\xrightarrow{R_3 \longleftrightarrow R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{array}\right] \\ &\xrightarrow{R_3 \longleftrightarrow R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right] \end{aligned}$$

The zero row on the left hand side (the $\mathbf{A}$-part) of the adjoined matrix shows that $\mathbf{A}$ cannot be row-reduced to the identity matrix. Hence, the matrix $\mathbf{A}$ has no inverse.