Math 3013
SOLUTIONS TO FIRST EXAM
11:30 - 12:20, February 18, 2022

1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Compute the matrix products **AB** and **BA** (if they exist)

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} (1)(2)+(-1)(0) & (1)(0)+(-1)(-1) & (1)(1)+(-1)(0) \\ (0)(2)+(3)(0) & (0)(0)+(3)(-1) & (0)(1)+(3)(0) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 0 \end{bmatrix} \end{aligned}$$

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$$\mathbf{BA} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \text{ is undefined since } \#Columns(\mathbf{B}) \neq \#Rows(\mathbf{A})$$

2. For each of the following augmented matrices, indicate
   (i) the number of equations and the number of variables in the corresponding linear system
   (ii) whether or not the corresponding linear system has a solution
   (iii) if the corresponding linear system does have a solution, the number of free variables in the solution.

Hint: Note that these augmented matrices are already in row echelon form and that you do not have to do any calculations to answer the questions.

(a) $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

   (i) 4 equations in 4 unknowns
   (ii) there is a solution
   (iii) Column 3 is the only column on the left without a pivot. Therefore, there will be 1 free parameter (x_3) in the solution.

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

   (i) 3 equations in 4 unknowns
   (ii) there is a solution
   (iii) Columns 2 and 4 are the only columns on the left without pivots. Therefore, there will be 2 free parameters (x_2 and x_3) in the solution.

(c) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

   (i) 4 equations in 3 unknowns
   (ii) There is no solution since the equation corresponding to the last row says $0 = 1$ (which implies there was a contradiction amongst the original equations).

3. Consider the following linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 1 \\ 2x_1 + x_2 + x_3 + 2x_4 &= -1 \\ -x_1 + x_2 + 2x_3 - x_4 &= 3 \end{aligned}$$

Write down the corresponding augmented matrix and row reduce it to row-echelon form.

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$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 & -1 \\ -1 & 1 & 2 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 3 & -3 & 0 & -3 \\ 0 & 0 & 4 & 0 & 4 \end{array}\right] \qquad (R.E.F.)$$

4. Row reduce the following augmented matrix to **reduced** row-echelon form.

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 & -2 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad (R.R.E.F.)$$

5. Suppose the augmented matrix below is the Reduced Row Echelon Form of an augmented matrix of a linear system. Display the solution of the linear system as a hyperplane (within the space of variables).

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

- The augmented matrix in R.R.E.F. has columns 2 and 4 on the left without pivots. This means that x_2 and x_4 are to be free variables in the solution. Converting back to equations, and moving the free variables to the right hand side yields

$$\left.\begin{array}{r} x_1 - 2x_4 = 1 \\ x_3 + x_4 = 2 \\ x_5 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = 1 + 2x_4 \\ x_3 = 2 - x_4 \\ x_5 = 0 \end{array}\right.$$

Next we write down a typical solution vector, substituting where we can for the fixed variables x_1, x_3, and x_5 :

$$\mathbf{x} = \left[\begin{array}{l} 1 + 2x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{array}\right] + x_2 \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right] + x_4 \left[\begin{array}{c} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{array}\right]$$

which displays the solution set as a 2-dimensional hyperplane in $\mathbb{R}^5$.

6. Determine if the matrix below has an inverse and, if so, determine its inverse. (Hint: a single row reduction process will answer both parts of to this problem.)

$$\mathbf{A} = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{array}\right]$$

- We form the adjoined matrix $[\mathbf{A} \mid \mathbf{I}]$ and row reduce it to its Reduced Row Echelon Form:

$$\begin{aligned}
[\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_3 \longleftrightarrow R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_2 \longleftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \\
&\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]
\end{aligned}$$

Since the $\mathbf{A}$-part of the reduced row echelon form of $[\mathbf{A} \mid \mathbf{I}]$ is the identity matrix, the right hand side (the $\mathbf{I}$-part) of the RREF is the inverse of $\mathbf{A}$. Thus,

$$\mathbf{A} = \left[\begin{array}{ccc} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 0 \end{array}\right]$$