Math 3013
SOLUTION TO FIRST EXAM
10:30 - 11:20pm, September 15, 2021

1. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Compute the matrix products **AB** and **BA** (if they exist).

- $$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2+0-1 & 0+0+1 & -2+0+1 \\ 1+0+0 & 0-2+0 & -1-1+0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & -2 \end{bmatrix} \end{aligned}$$

BA is undefined since the # of columns of the first factor $\neq$ # columns of the second factor

2. For each of the following augmented matrices, indicate

- the number of equations and the number of variables in the original linear system
- whether or not the corresponding linear system has a solution
- if the corresponding linear system does have a solution, the number of free variables in the solution.

Hint: note that these augmented matrices are already in row echelon form.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

- 3 equations in 4 variables
  no solution (because of the pivot in last column of last row)

(b) $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- 4 equations in 5 variables
  there is a solution
  two free variables, x_3 and x_5 (because there are no pivots in the corresponding columns on the L.H.S.)

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$

- 3 equation in 4 variables
  there is a solution
  1 free variable, x_2, because on the L.H.S. only the second column lacks a pivot

3. Consider the following linear system

$$\begin{aligned} 3x_2 + x_3 &= 0 \\ x_1 - x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

Write down the corresponding augmented matrix and row reduce it to row-echelon form.

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$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = R.E.F.\,([\mathbf{A}|\mathbf{b}])$$

4. Row reduce the following augmented matrix to **reduced** row-echelon form.

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccccc|c} 2 & 0 & 2 & 6 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\xrightarrow[R_3 \to -\frac{1}{3}R_3]{R_1 \to \frac{1}{2}R_1,\; R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1- \to R_1 - R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1- \to R_1 - R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = R.R.E.F.\,([\mathbf{A}|\mathbf{b}])$$

5. Suppose the augmented matrix below is the Reduced Row Echelon Form of an augmented matrix of a linear system. Display the solution of the linear system as a hyperplane (within the space of variables).

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

- Columns 2 and 4 of the R.R.E.F. do not contain pivots, and so x_2 and x_4 will be free variables in the solution. Writing down the equations corresponding to this augmented matrix, and moving the free variables to the right hand side we have

$$\left.\begin{array}{r} x_1 + 2x_4 = 1 \\ x_3 - x_4 = 2 \\ x_5 = 2 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = 1 - 2x_4 \\ x_3 = 2 + x_4 \\ x_5 = 2 \end{array}\right.$$

  And so a typical solution vector will be of the form

$$\mathbf{x} = \left[\begin{array}{c} 1 - 2x_4 \\ x_2 \\ 2 + x_4 \\ x_4 \\ 2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \\ 2 \\ 0 \\ 2 \end{array}\right] + x_2 \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right] + x_4 \left[\begin{array}{c} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}\right]$$

  which displays the solution vectors as points on a 2-dimensional plane.

6. Determine if the following matrix has an inverse. If the matrix does have an inverse, determine its inverse. (Hint: a single row reduction process can determine the answer to both parts of this problem.)

$$\mathbf{A} = \left[\begin{array}{ccc} -1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

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$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to -R_3]{R_1 \to -R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & -1 & -1 \end{array}\right] = R.R.E.F.\,([\mathbf{A}|\mathbf{I}])$$

  Since the left hand side of the R.R.E.F. of $[\mathbf{A}|\mathbf{I}]$ is the identity matrix, the right hand side must be $\mathbf{A}^{-1}$. Thus,

$$\mathbf{A}^{-1} = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & -1 \\ -2 & -1 & -1 \end{array}\right]$$