Math 3013
SOLUTIONS TO FIRST EXAM
12:30 - 1:20pm, September 15, 2021

1. Let
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$
Compute the matrix products **AB** and **BA** (if they exist)

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AB : product is undefined since #columns of the first factor $\neq$ #rows of the second factor

$$\begin{aligned}\mathbf{BA} &= \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+0+2 & 0+0+1 & -2+0+1 \\ 0+0+0 & 0-1+0 & 0+0+0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\end{aligned}$$
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2. For each of the following augmented matrices, indicate

- the number of equations and the number of variables in the corresponding linear system
- whether or not the corresponding linear system has a solution
- if the corresponding linear system does have a solution, the number of free variables in the solution.

Hint: note that these augmented matrices are already in row echelon form and you do not have to actually solve the systems.

(a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

- 4 equations and 3 unknowns
  no solution (since the last row corresponds to the equation $0 = 1$ which is a contradiction)

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- 3 equations in 4 variables
  solutions exist
  2 free variables, x_2 and x_4, since columns 2 and 4 on the left hand side of the R.E.F. don't have pivots

(c) $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right]$

- 3 equation in 3 variables
  there is a solution
  no free variables in solution, since the left hand side of the R.E.F. does not have any columns without pivots

3. Consider the following linear system

$$\begin{aligned} x_2 + 2x_3 + x_4 &= 1 \\ 2x_1 + x_2 + x_3 + 2x_4 &= -1 \\ x_1 + x_2 + 2x_3 - x_4 &= 3 \end{aligned}$$

Write down the corresponding augmented matrix and row reduce it to row-echelon form.

- $$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 & -1 \\ 1 & 1 & 2 & -1 & 3 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & 2 & -1 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 3 \\ 0 & -1 & -3 & 4 & -7 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 3 \\ 0 & -1 & -3 & 4 & -7 \\ 0 & 0 & -1 & 5 & -6 \end{array}\right]$$

$$= R.R.E.F.\,([\mathbf{A}|\mathbf{b}])$$

4. Row reduce the following augmented matrix to **reduced** row-echelon form.

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

- $$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 6 & -2 \\ 0 & 0 & 1 & 0 & 6 & -2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 & -2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$= R.R.E.F.\,([\mathbf{A}|\mathbf{b}])$$

5. Suppose the augmented matrix below is the Reduced Row Echelon Form of an augmented matrix of a linear system. Display the solution of the linear system as a hyperplane (within the space of variables).

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

- Columns 2 and 4 of $R.R.E.F.\left([\mathbf{A}|\mathbf{b}]\right)$ do not contain pivots, and so x_2 and x_4 will be free variables in the solution. Converting the augmented matrix back to equations, and moving the free variables to the right hand side we get

$$\left.\begin{array}{r} x_1 - 2x_4 = 1 \\ x_3 + x_4 = 2 \\ x_5 = 1 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = 1 + 2x_4 \\ x_3 = 2 - x_4 \\ x_5 = 1 \end{array}\right.$$

Thus, a typical solution vector will have the form

$$\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array}\right] = \left[\begin{array}{c} 1 + 2x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{array}\right] + x_2 \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right] + x_4 \left[\begin{array}{c} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{array}\right]$$

which displays the solution vectors as living on a 2-dimensional plane.

6. Determine if the matrix below has an inverse and, if so, determine its inverse. (Hint: a single row reduction process will answer both parts of to this problem.)

$$\mathbf{A} = \left[\begin{array}{ccc} 1 & -1 & 1 \\ -2 & 2 & -1 \\ 1 & 0 & 0 \end{array}\right]$$

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$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -2 & 2 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

Since $R.R.E.F.\left([\mathbf{A}|\mathbf{I}]\right)$ has the identity matrix on its left hand side, its right hand side must be $\mathbf{A}^{-1}$. Thus

$$\mathbf{A}^{-1} = \left[\left|\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{array}\right.\right]$$