Math 3013.25578
SOLUTIONS TO SECOND EXAM
March 28, 2022

1. Define, precisely, the following notions (where V, W are to be regarded as general vector spaces). space V).

(a) (5 pts) a **subspace** of V is

- a subset W of V that is closed under both scalar multiplication and vector addition; i.e.,
  - For all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in W$, $\lambda \mathbf{v} \in W$
  - For all $\mathbf{v}_1, \mathbf{v}_2 \in W$, $\mathbf{v}_1 + \mathbf{v}_2 \in W$

(b) (5 pts) a **basis** for a vector space V is

- a set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\} \subset V$ such that every vector $\mathbf{v} \in V$ can be uniquely expressed as
$$\mathbf{v} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_k \mathbf{b}_k$$

(c) (5 pts) a **set of linearly independent vectors** in V is

- a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ such that the only solution of
$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k = \mathbf{0}$$
is $c_1 = 0$, $c_2 = 0$, $\ldots, c_k = 0$

(d) (5 pts) a **linear transformation** from a vector space V to vector space W is

- a function $T : V \to W$ such that
  - $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\mathbf{x} \in V$
  - $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in V$

2. (10 pts) Prove or disprove that the points on the circle $S = \{[x, y] \in \mathbb{R}^2 \mid y = x + 2\}$ is a subspace of $\mathbb{R}^2$.

- A subspace has to be closed under both scalar multiplication and vector addition.
  - *closure under scalar multiplication.* Consider $\mathbf{v} = [0, 2] \in S$ and $\lambda = 0 \in \mathbb{R}$. Then
$$\lambda \mathbf{v} = [0 \cdot 0, 0 \cdot 2] = [0, 0] \notin S \text{ since } 0 \neq 0 + 2$$
    So S is not closed under scalar multiplication. Hence S is not a subspace.
  - *closure under vector addition.* Let $\mathbf{v}_1 = [0, 2]$, $\mathbf{v}_2 = [1, 3]$, so that $\mathbf{v}_1, \mathbf{v}_2 \in S$. Then
$$\mathbf{v}_1 + \mathbf{v}_2 = [0 + 2, 1 + 2] = [2, 3] \notin S \text{ since } 3 \neq 2 + 2$$
    So S is not closed under vector addition. Hence S is not a subspace

(Either one of these arguments is a satisfactory solution.)

3. (10 pts) Let $W = span\left([1,1,1],[1,-2,1],[3,0,3]\right) \subset \mathbb{R}^3$. Find a basis for W.

- Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 0 & 3 \end{bmatrix}$$

Then

$$span\left(W\right) = RowSp\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 0 & 3 \end{bmatrix}\right) \xrightarrow{\text{row reduction}} RowSp\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$\begin{aligned} \text{basis for } W &= \text{basis for } RowSp\left(\mathbf{A}\right) \\ &= \text{nonzero rows of R.E.F.}\left(\mathbf{A}\right) \\ &= \left\{[1,0,1],[0,1,0]\right\} \end{aligned}$$

4. Consider the following matrix: $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 2 & 2 & 0 & 3 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

(a) (10 pts) Row reduce this matrix to reduced row echelon form

- $$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 2 & 2 & 0 & 3 \\ 0 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (5 pts) Find a basis for the row space of $\mathbf{A}$.

- A basis for the row space of the matrix $\mathbf{A}$ is formed by the non-zero rows of any row echelon form of $\mathbf{A}$. Thus

$$\text{basis for } RowSp\left(\mathbf{A}\right) = \left\{[1,0,0,1],[0,0,1,1/2]\right\}$$

(c) (5 pts) Find a basis for the column space of $\mathbf{A}$.

- A basis for the column space of $\mathbf{A}$ is formed by the columns of $\mathbf{A}$ corresponding to the columns of a row echelon form of $\mathbf{A}$ which contain pivots. Since the first and second columns of the RREF of $\mathbf{A}$ are where the pivots of the RREF reside, the first and second columns of $\mathbf{A}$ will be a basis for the column space of $\mathbf{A}$:

$$\text{basis for } ColSp\left(\mathbf{A}\right) = \left\{\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}\right\}$$

(d) (5 pts) Find a basis for the null space of $\mathbf{A}$.

- To find a basis for the null space of $\mathbf{A}$, we must solve $\mathbf{Ax} = \mathbf{0}$. From the reduced row echelon form of $\mathbf{A}$, we conclude that if $\mathbf{x} = [x_1, x_2, x_3, x_4]$ is a solution of $\mathbf{Ax} = \mathbf{0}$, then

$$\left.\begin{array}{r} x_1 + 0 + 0 + x_4 = 0 \\ 0 + x_2 + 0 + \frac{1}{2}x_4 = 0 \\ 0 + 0 + 0 + 0 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = -x_4 \\ x_2 = -\frac{1}{2}x_4 \end{array}\right. \Rightarrow \mathbf{x} = \begin{bmatrix} -x_4 \\ -\frac{1}{2}x_4 \\ x_3 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

and so

$$\text{basis for } NullSp\left(\mathbf{A}\right) = \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right\}$$

(e) (5 pts) What is the rank of $\mathbf{A}$?

- $rank\left(A\right) = \dim\left(RowSp\left(A\right)\right) = \dim\left(ColSp\left(\mathbf{A}\right)\right) = 2$

5. (10 pts) Let $\mathbf{A}$ be an $n \times m$ matrix. Show that the function $T_{\mathbf{A}} : \mathbb{R}^m \to \mathbb{R}^n : T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ (via matrix multiplication on the right) is a linear transformation

(i) *Compatibility with scalar multiplication.* Suppose $\mathbf{x} \in \mathbb{R}^m$. The is a solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$, then

$$T(\lambda\mathbf{x}) \equiv \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{A}\mathbf{x}) = \lambda T(\mathbf{x}) \quad \text{(scalar multiplication commutes with matrix multiplication)}$$

and so T is compatible with scalar multiplication.

(ii) *Compatibility with vector addition.* Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$. Then

$$T(\mathbf{x}_1 + \mathbf{x}_3) = \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 = T(\mathbf{x}_1) + T(\mathbf{x}_2)$$

and so T is compatible with vector addition.

- Since T is a function between to vector spaces that is compatible with both scalar multiplication and vector addition, T is a linear transformation.

6. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 + x_3, x_2 - x_3]$.

(a) (10 pts) Find the matrix $\mathbf{A}_T$ representating T:

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$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Note that $\mathbf{A}_T$ is already in R.R.E.F.

(b) (5 pts) Find a basis for $Range(T) \equiv \{\mathbf{y} \in \mathbb{R}^2 \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^4\}$.

- We have $range(T) = ColSp(\mathbf{A}_T)$. Note that $\mathbf{A}_T$ is already in row echelon form and it has pivots in columns 1 and 2. Therefore, the first two columns of $\mathbf{A}_T$ will be basis vectors. Thus, $\{[1,0], [0,1]\}$ will be a basis for $Range(T)$.

(c) (5 pts) Find a basis for $ker(T) \equiv \{\mathbf{x} \in \mathbb{R}^3 \mid T(\mathbf{x}) = \mathbf{0}\}$

- We have $\ker(T) = NullSp(\mathbf{A}_T) =$ solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$. Since the matrix $\mathbf{A}_T$ is already in R.R.E.F. we can rapidly write down the solutions of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$:

$$\left.\begin{array}{r} x_1 + 0 + x_3 = 0 \\ 0 + x_2 - x_3 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{c} x_1 = -x_3 \\ x_2 = x_3 \end{array}\right.$$

$$\Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and so

$$\text{basis for } Ker(T) = \text{basis for } NullSp(\mathbf{A}_T) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$