Math 3013.62667
SOLUTIONS TO SECOND EXAM
October 22, 2021

1. Complete the following mathematical definitions

(a) (5 pts) A **subspace** of a vector space V is ...

- a subset W of V such that
  (i) $\lambda \in \mathbb{R}$, $\mathbf{w} \in W \quad \Rightarrow \quad (\lambda \mathbf{w}) \in W$
  (ii) $\mathbf{w}_1, \mathbf{w}_2 \in W \quad \Rightarrow \quad (\mathbf{w}_1 + \mathbf{w}_2) \in W$

(b) (5 pts) A **basis** for a subspace W is ...

- a set of vectors $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ such that every vector $\mathbf{w} \in W$ can be expressed as

$$\mathbf{w} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k$$

  for exactly one choice of coefficients $c_1, c_2, \dots, c_k$.

(c) (5 pts) A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is **linearly independent** if ...

- the only solution of

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

  is

$$x_1 = 0 \ , \ x_2 = 0 \ , \ \dots \ , \ x_k = 0$$

(d) (5 pts) A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a **linear transformation** if ...

- (i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^m$
  (ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

2. Consider the vectors $\{[0,1,-1,2],[1,1,1,1],[2,1,3,0]\} \in \mathbb{R}^4$

(a) (5 pts) Determine if these vectors are linearly independent.

- We'll form a matrix using the given vectors as rows, and then row reduce that matrix to Row Echelon Form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we ended up with a zero row, the original set of vectors **are not linearly independent**.

(b) (5 pts) What is the dimension of the subspace generated by these vectors?

- The row space of the matrix $\mathbf{A}$ constructed in part (a) has exactly two basis vectors ($[1,1,1,1]$ and $[0,1,-1,2]$) and so

$$W = span\left([0,1,-1,2],[1,1,1,1],[2,1,3,0]\right) = RowSp\left(\mathbf{A}\right)$$

is 2-dimensional.

3. Given that the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 2 & 0 & 2 & 5 \\ 1 & 0 & 1 & 3 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) (5 pts) Find a basis for the row space of $\mathbf{A}$.

- A basis for $RowSp\left(\mathbf{A}\right)$ is given by the non-zero rows of $R.E.F.\left(\mathbf{A}\right)$. Hence,

$$\text{basis for } RowSp\left(\mathbf{A}\right) = \{[1,0,1,0], [0,0,0,1]\}$$

(b) (5 pts) Find a basis for the column space of $\mathbf{A}$.

- A basis for $ColSp\left(\mathbf{A}\right)$ is given by the columns of $\mathbf{A}$ that correspond to the columns of $R.E.F.\left(\mathbf{A}\right)$ that contain pivots.

  Since the pivots of $R.E.F.\left(\mathbf{A}\right)$ occur in the 1^{st} and 4^{th} columns,

$$\text{basis for } ColSp\left(\mathbf{A}\right) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \right\}$$

(c) (5 pts) Find a basis for the null space of $\mathbf{A}$.

- We need to solve $\mathbf{Ax} = \mathbf{0}$. From the row reduction of $\mathbf{A}$ we can infer

$$[\mathbf{A}|\mathbf{0}] \quad \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \left\{ \begin{array}{r} x_1 + x_3 = 0 \\ x_4 = 0 \\ 0 = 0 \end{array} \right.$$

Since columns 2 and 3 of the R.R.E.F. of $[\mathbf{A}|\mathbf{0}]$ don't have pivots, x_2 and x_3 will be the free variable of the solution:

$$\mathbf{x} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \text{basis for } NullSp\left(\mathbf{A}\right) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(d) (5 pts) What is the rank of $\mathbf{A}$?

- Since the row space (and column space) of $\mathbf{A}$ has 2 basis vectors

$$Rank\left(\mathbf{A}\right) = 2$$

4. Consider the following linear transformation:
$T : \mathbb{R}^3 \to \mathbb{R}^3 : T([x_1, x_2, x_3]) = [x_1 + x_2, x_1 - x_3, x_1 + 2x_2 + x_3]$.
(a) (5 pts) Find the matrix $\mathbf{A}_T$ such that $\mathbf{A}_T \mathbf{x} = T(\mathbf{x})$.

- $$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) (5 pts) Find a basis for the range of T.

- Let's first row reduce $\mathbf{A}_T$ to its R.R.E.F.

$$\mathbf{A}_T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to -R_2]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R.R.E.F.(\mathbf{A}_T)$$

Since $Range(T) = ColSp(\mathbf{A}_T)$ and a basis for $ColSp(\mathbf{A}_T)$ is given by the first two columns of $\mathbf{A}_T$ (as these are the columns of $\mathbf{A}_T$ that correspond to the columns of $R.R.E.F.(\mathbf{A}_T)$ that have pivots).

$$\text{basis for } Range(T) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

(c) (5 pts) Find a basis for the kernel of T.

- We have $\ker(T) = NullSp(\mathbf{A}_T) =$ solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$. From the R.R.E.F. of $\mathbf{A}_T$, we see that solutions of $\mathbf{A}_T\mathbf{x} = 0$ must satisfy

$$\left.\begin{array}{r} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x_1 = x_3 \\ x_2 = -x_3 \end{array}\right. \Rightarrow \mathbf{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and so

$$\text{basis for } \ker(T) = \{[1, -1, 1]\}$$

5. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$

(a) 7 pts) Compute $\det(\mathbf{A})$ via a cofactor expansion along the third row.

- $$\begin{aligned} \det(\mathbf{A}) &= (1)(-1)^{3+1}\det(\mathbf{M}_{13}) + 0 + 0 + 0 \\ &= (1)(-1)^4 \det\left(\begin{bmatrix} 0 & 0 & 3 \\ 1 & -3 & 0 \\ 0 & -2 & 0 \end{bmatrix}\right) \\ &= \left(0 + 0 + (3)(-1)^{1+3}\det\left(\begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}\right)\right) \\ &= (3)(-1)^4(-2+0) \\ &= -6 \end{aligned}$$

(b) (8 pts) Compute $\det(\mathbf{A})$ by row reducing $\mathbf{A}$ to an upper triangular matrix.

- $$\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_3 \longleftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = R.E.F.(\mathbf{A})$$

  Since we performed 2 row interchanges, and 0 row rescalings, in our row reduction to R.E.F.,

  $$\det(\mathbf{A}) = (-1)^2 \det(R.E.F.(\mathbf{A})) = \det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right) = (1)(1)(3)(-2) = -6$$

6. (10 pts) Use Cramer's Rule to solve

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 + x_2 &= 3 \end{aligned}$$

- We have

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{A}) = (1)(1) - (2)(-1) = 3 \\ \mathbf{B}_1 &= \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{B}_1) = (3)(1) - (2)(3) = -3 \\ \mathbf{B}_2 &= \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{B}_2) = (1)(3) - (3)(-1) = 6 \end{aligned}$$

Hence, Cramer's Rule tells us that the components of a solution vector are

$$\begin{aligned} x_1 &= \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{-3}{3} = -1 \\ x_2 &= \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = \frac{6}{3} = 2 \end{aligned}$$

7. (10 pts) Find all the cofactors of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ and then use these cofactors to compute $\mathbf{A}^{-1}$

- The cofactors are given by the formula $c_{ij} = (-1)^{i+j} \det(\mathbf{M}_{ij})$ where $\mathbf{M}_{ij}$ is the 1×1 matrix obtained from $\mathbf{A}$ by deleting its i^{th} row and j^{th} column.

$$\begin{aligned} c_{11} &= (-1)^{1+1} \det([4]) = 4 \\ c_{12} &= (-1)^{1+2} \det([3]) = -3 \\ c_{21} &= (-1)^{2+1} \det([3]) = -2 \\ c_{22} &= (-1)^{2+2} \det([2]) = 2 \end{aligned}$$

So the cofactor matrix is

$$\mathbf{C} = \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix}$$

We can now apply the cofactor formula for $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \frac{1}{(8-6)} \begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$$