Math 3013
SECOND EXAM
November 3, 2020

1. Consider the vectors $\{[1,2,1,1,0]\ ,\ [0,1,2,1,1]\ ,\ [-1,-1,1,0,1]\ ,\ [-1,0,3,1,2]\} \in \mathbb{R}^5$

(a) (10 pts) Determine if these vectors are linearly independent.

- Writing these vectors as the rows of a matrix $\mathbf{A}$, we can determine if the vectors are linearly independent by checking if the R.E.F. of $\mathbf{A}$ has any non-zero row vectors. If R.E.F.($\mathbf{A}$) has any zero row vectors, then the original set of vectors are linearly dependent.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & -3 & -1 & -2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R.R.E.F.\left(\mathbf{A}\right)$$

Since we have two zero row vectors, the original set of matrices are **not linearly independent**.

(b) (5 pts) What is the dimension of the subspace generated by these vectors?

- The row reduction calculation in part (a) also produces a basis for the row space of $\mathbf{A}$, which is also a basis for the subspace generated by the original set of vectors. Since we have two basis vectors, the dimension of the subspaces is 2.

2. Write the definitions (as stated in class) of the following notions. (5 pts each)

(a) A **subspace** W of $\mathbb{R}^n$.

- A *subspace* W of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ such that
  - $\lambda \in \mathbb{R}\ ,\ \mathbf{w} \in W \quad \Rightarrow \quad (\lambda \mathbf{w}) \in W$
  - $\mathbf{w}_1, \mathbf{w}_2 \in W \quad \Rightarrow \quad (\mathbf{w}_1 + \mathbf{w}_2) \in W$

(b) A **basis** for a subspace W of $\mathbb{R}^n$.

- A *basis* for a subspace W of $\mathbb{R}^n$ is a set of vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ such that
  - every vector $\mathbf{w} \in W$ can be expressed as

(*) $$\mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k$$

  - for each $\mathbf{w} \in W$, the coefficients $c_1, \dots, c_k$ in (*) are unique.

(c) A set of **linearly independent vectors**

- A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are *linearly independent* if the only solution of

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

is

$$x_1 = 0\ ,\ x_2 = 0\ ,\ \cdots\ ,\ x_k = 0$$

(d) A **linear transformation**

- A *linear transformation* is a function $T : V \to W$ between two vector spaces such that
  - $\lambda \in \mathbb{R}\ ,\ \mathbf{v} \in V \quad \Rightarrow \quad T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$
  - $\mathbf{v}_1, \mathbf{v}_2 \in V \quad \Rightarrow \quad T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$

3. Given that the matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 & 0 & 2 \\ -1 & -2 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) (5 pts) Find a basis for the row space of $\mathbf{A}$.

- The non-zero rows of $R.E.F.(\mathbf{A})$ provide a basis for the row space of $\mathbf{A}$:

$$\text{basis for row space} = \{[1,2,0,1]\ ,\ [0,0,1,-2]\}$$

(b) (5 pts) Find a basis for the column space of $\mathbf{A}$.

- The columns of $\mathbf{A}$ that correspond to the columns of its R.E.F. that contain pivots will provide a basis for the column space of $\mathbf{A}$:

$$\text{basis for column space} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) (5 pts) Find a basis for the null space of $\mathbf{A}$.

- The null space of $\mathbf{A}$ is the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$. We are given the R.R.E.F. of the coefficient matrix $\mathbf{A}$, so we can immediately write down the solutions of $\mathbf{Ax} = \mathbf{0}$:

$$\left.\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ x_3 - 2x_4 &= 0 \\ 0 &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = -2x_2 - x_4 \\ x_2 \text{ is a free variable} \\ x_3 = 2x_4 \\ x_4 \text{ is a free variable} \end{array}\right.$$

$$\mathbf{x} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

and

$$\text{basis for } NullSp(\mathbf{A}) = \{[-2,1,0,0]\ ,\ [-1,0,2,1]\}$$

(d) (5 pts) What is the rank of $\mathbf{A}$?

- The rank of $\mathbf{A}$ is the common dimension of its row and column spaces. Since both the row and column spaces have two basis vectors

$$\text{rank}(\mathbf{A}) = 2$$

4. Consider the following linear transformation:
$T : \mathbb{R}^3 \to \mathbb{R}^3 : T([x_1\ ,\ x_2\ ,\ x_3) = [x_1 + 2x_2 + x_3\ ,\ x_1 + x_2\ ,\ x_1 - x_3]$.

(a) (10 pts) Find a matrix that represents T.

- We have

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

(b) (5 pts) Find a basis for the range of T.

- We have $Range(T) = ColSp(\mathbf{A}_T)$. Thus, we find a basis for $Range(T)$ by finding a basis for the column space of $\mathbf{A}_T$

$$\mathbf{A}_T = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R.R.E.F.(\mathbf{A}_T)$$

Grabbing the columns of $\mathbf{A}_T$ corresponding to the columns of its R.R.E.F. that have pivots we find

$$\text{basis for Range}\,(T) = \{[1,1,1]\ ,\ [2,1,0]\}$$

(c) (5 pts) Find a basis for the kernel of T.

- – $\ker(T) = NullSp\,(\mathbf{A}_T) =$ solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$. Using the R.R.E.F. of $\mathbf{A}_T$ that we just calculated we find the solutions of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ must be vectors of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and so

$$\text{basis for } \ker(T) = \{[1,-1,1]\}$$

5. (15 pts) Use cofactor expansions to compute the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

- The second column has only one non-zero entry. So we'll carry out a cofactor expansion along the second column of $\mathbf{A}$

$$\begin{aligned} \det(\mathbf{A}) &= 0 + 0 + (2)(-1)^{3+2} \det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \\ &= (2)(-1)\left(0 + (1)(-1)^{2+2} \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + 0\right) \\ &= (2)(-1)(1)(1)(2-1) \\ &= -2 \end{aligned}$$

6. (15 pts) Use the row reduction method to determine the determinant of $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- The following sequence of elementary row operations converts $\mathbf{A}$ to row echelon form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have

$$\det(\mathbf{A}) = (-1)^r \frac{1}{\lambda_1 \dots \lambda_k} \det(R.E.F.(\mathbf{A}))$$

Here r is the number of row interchanges used in the row reduction, and $\lambda_1, \dots, \lambda_k$ are the row rescalings used. In our case, $r = 1$ and we didn't use any row rescalings. Hence,

$$\begin{aligned} \det(\mathbf{A}) &= (-1) \det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = (-1)(1)(2)(-1)(1) \\ &= 2 \end{aligned}$$