Math 3013 Linear Algebra
Solutions to Final Exam
May 4, 2022

1. Complete the following definitions

(a) (5 pts) A **subspace** of a vector space V is ...

- a subset W of V such that
  (i) whenever $\lambda \in \mathbb{R}$, $\mathbf{w} \in W$, $(\lambda \mathbf{w}) \in W$
  (ii) whenever $\mathbf{w}_1, \mathbf{w}_2 \in W$, $(\mathbf{w}_1 + \mathbf{w}_2) \in W$

(b) (5 pts) A **basis** for a subspace W of a vector space V is ...

- a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ such that
  (i) $W = span\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$
  (ii) If $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, then coefficients $c_1, \dots, c_k$ are unique.

(c) (5 pts) A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a **linearly independent** set of vectors if ...

- the only solution of $x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$ is $x_1 = 0, \dots, x_k = 0$.

(d) (5 pts) A **linear transformation** between two vector spaces V and W is ...

- A linear transformation between two vector space V and W is a function $T : V \to W$ such that
  (i) $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in V$
  (ii) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

2. Suppose each of the following augmented matrices is a Row Echelon Form of the augmented matrix of a linear system $\mathbf{Ax} = \mathbf{b}$. Describe the original system (i.e., how many equations in how many unknowns) and describe the solution space of the corresponding linear system. (Determine if there are solutions; and, if there are solutions, how many free parameters are needed to describe the general solution.)

(a) (5 pts) $\left[\begin{array}{ccccc|c} 0 & -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$: There are solutions. The coefficient matrix side of augmented matrix in REF has 2 columns without pivots, so there will be 2 free parameters in the general solution.

(b) (5 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$: There are no solutions, as the equation corresponding to the second to last row is $0 = 1$, a mathematical contradiction.

(c) (5 pts) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$: There is a solution. Since there are no columns without pivots on the coefficient matrix side of the augmented matrix, there are no free parameters in the general solution (in other words, there is a unique solution).

3. (10 pts) Solve the following linear system, expressing the solution set as a hyperplane.

$$\begin{aligned} x_1 - x_2 + x_4 &= 1 \\ x_1 + x_2 + x_4 &= 1 \\ x_1 + x_4 &= 1 \end{aligned}$$

- First, we row reduce the augmented matrix for the system to Reduced Row Echelon Form

$$[\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = RREF\left([\mathbf{A}|\mathbf{b}]\right)$$

  Then, we covert back to equations noting that x_3 and x_4 will be free variables in the solution (RREF has no pivots in columns 3 and 4).

$$\left.\begin{aligned} x_1 + x_4 &= 1 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} x_1 &= 1 - x_4 \\ x_2 &= 0 \end{aligned}\right.$$

  Finally, we write down the form of a solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

  which displays the solution set as a 2-dimensional hyperplane.

4. (10 pts) Compute the inverse of $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

- $$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] = \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$$
  and so
$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

5. (10 pts) Let $W = \{[x, y] \in \mathbb{R}^2 \mid x + 2y = 3\}$. Prove or disprove that W is a subspace of $\mathbb{R}^2$.

- Consider $\mathbf{w} = [1, 1]$. This vector is in W since $(1) + (2)(1) = 3$. Now consider the scalar multiple of $\mathbf{w}$ by the number 0.
$$(0)\,\mathbf{w} = [0, 0] \notin W \text{ since } (1)(0) + (2)(0) = 0 \neq 3$$
  Thus, W is not closed under scalar multiplication, and so W is not a subspace.

6. Consider the vectors $\{[1,1,1,1],[1,2,-1,1],[2,3,0,2],[3,4,1,3]\} \in \mathbb{R}^4$

(a) (10 pts) Determine if these vectors are linearly independent.

- We'll answer this question by row reducing the associated matrix formed by using the given vectors as rows

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 3 & 4 & 1 & 3 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  Since there is a zero row at the bottom of the matrix in RREF, the original row vectors could have been linearly independent. Thus, **the vectors are not linearly independent.**

(b) (5 pts) What is the dimension of the subspace generated by these vectors (i.e. the subspace spanned by these vectors)?

- The subspace generated by the original four vectors coincides with the row space of the matrix **A**. We see that there are two non-zero rows in $RREF(\mathbf{A})$, hence, $RowSp(\mathbf{A}) = span([1,1,1,1],[1,2,-1,1],[2,3,0,2],[3,4,1,3])$ is 2-dimensional.

7. Consider the following linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2.x_3]) = [x_2 - x_1 \, , \, x_1 - x_2]$.

(a) (10 pts) Find a matrix that represents T.

- $$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(b) (10 pts) Find a basis for the kernel of T (i.e. the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$).

- $$\begin{aligned} Ker(T) &= NullSp(\mathbf{A}_T) \\ &= NullSp\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\right) \\ &= NullSp\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \end{aligned}$$

  Converting back to equations

$$\mathbf{x} \in NullSp\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \quad \Rightarrow \quad \begin{cases} x_1 - x_2 = 0 \\ 0 = 0 \end{cases}$$

$$\Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  and so

$$\text{basis for } Ker(T) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

8. (a) (10 pts) Find the eigenvalues and the eigenvectors of the following matrix : $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

- We have

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2(-\lambda)$$

So the eigenvalues of $\mathbf{A}$ are 2 and 0. Since we have two factors of $(2-\lambda)$ in $p_{\mathbf{A}}(\lambda)$ the eigenvalue $\lambda = 2$ has algebraic multiplicity 2. Since we have only one factor of $(0-\lambda)$ in $p_{\mathbf{A}}(\lambda)$, the eigenvalue $\lambda = 0$ has algebraic multiplicity 1.

2-eigenspace

$$E_2 = NullSp(\mathbf{A} - (2)\mathbf{I}) = NullSp\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Since the 2-eigenspace of $\mathbf{A}$ has only one basis vector, the geometric multiplicity of the eigenvalue 2 is 1.

0-eigenspace

$$E_0 = NullSp(\mathbf{A} - (0)\mathbf{I}) = NullSp\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= span\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right)$$

Since the 0-eigenspace of $\mathbf{A}$ has only one basis vector, the geometric multiplicity of the eigenvalue 0 is 1.

(b) (5 pts) What are the algebraic multiplicities and geometric multiplicities of the eigenvalues of $\mathbf{A}$?

- We have

eigenvalue	basis for eigenspace	alg. mult.	geom. mult.
2	$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$	2	1
0	$\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right\}$	1	1

(c) (5 pts) Is this matrix diagonalizable?

- No. We need 3 linearly independent eigenvectors in order to diagonalize a 3×3 matrix. But we only found two linearly independent eigenvectors; and so $\mathbf{A}$ is not diagonalizable.

9. (15 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

- First, we need to find the eigenvalues and eigenvectors of $\mathbf{A}$.

$$p_{\mathbf{A}}(\lambda) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

So the eigenvalues of $\mathbf{A}$ are 2 and 4. Next, we look for the corresponding eigenvectors:

2-eigenspace

$$\begin{aligned} E_2 &= NullSp\left(\begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

4-eigenspace

$$\begin{aligned} E_4 &= NullSp\left(\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

We use the eigenvalues of $\mathbf{A}$ to construct the diagonal matrix $\mathbf{D}$ and then use the corresponding eigenvectors as the columns of the matrix $\mathbf{C}$. Thus,

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

10. Let $W = span\left([1\ ,1,1]\ ,\ [0,1,1]\right) \subset \mathbb{R}^3$.

(a) (7 pts) Find the orthogonal complement $W_\perp = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{x} \in W\}$ of W in $\mathbb{R}^3$.

- Let $\mathbf{b}_1 = [1,1,1]$, $\mathbf{b}_2 = [0,1,1]$. Every vector in $W_\perp$ must perpendicular to both of these basis vectors.

$$\left.\begin{array}{c} \mathbf{b}_1\cdot\mathbf{x} = 0 \\ \mathbf{b}_2\cdot\mathbf{x} = 0 \end{array}\right\} \quad \Rightarrow \quad \left[\begin{array}{ccc} \leftarrow & \mathbf{b}_1 & \rightarrow \\ \leftarrow & \mathbf{b}_2 & \rightarrow \end{array}\right]\mathbf{x} = \mathbf{0}$$

$$\Rightarrow \quad W_\perp = NullSp\left(\left[\begin{array}{ccc} \leftarrow & \mathbf{b}_1 & \rightarrow \\ \leftarrow & \mathbf{b}_2 & \rightarrow \end{array}\right]\right) = NullSp\left(\left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right]\right) = NullSp\left(\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]\right)$$

$$= \text{solution set of } \left\{\begin{array}{r} x_1 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} = x_3\left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]$$

And so

$$W_\perp = span\left(\left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]\right)$$

(b) (8 pts) Let $\mathbf{v} = [1,-2,0]$, find the orthogonal decomposition $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$; $\mathbf{v}_W \in W$, $\mathbf{v}_\perp \in W_\perp$, of $\mathbf{v}$ with respect to W.

- Combining the basis for W with the basis for $W_\perp$, we have the following basis for $\mathbb{R}^3$

$$B = \{[1,1,1]\,,[0,1,1]\,,[0,-1,1]\}$$

We'll now find suitable coefficients, c_1, c_2, c_3 so that

$$c_1\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + c_2\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right] + c_3\left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ -2 \\ 0 \end{array}\right]$$

To solve this linear system, we row reduce the corresponding augmented matrix to RREF:

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & 1 & 1 & 0 \end{array}\right] \quad \longrightarrow \quad \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So we must have

$$c_1 = 1 \quad , \quad c_2 = -2 \quad , \quad c_3 = 1$$

and so

$$\mathbf{v} = (1)\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + (-2)\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right] + (1)\left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]$$

$$= \mathbf{v}_W + \mathbf{v}_\perp$$

where

$$\mathbf{v}_W = (1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \in W$$

$$\mathbf{v}_\perp = (1)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \in W_\perp$$