Math 3013.62979
Solutions to Final Exam
2:00pm – 4:00pm, December 8, 2020

1. Give the definitions of the following linear algebraic concepts:

(a) (5 pts) a **subspace** of a vector space V.

- A subspace of vector space V is a subset W of V such that
  (i) whenever $\lambda \in \mathbb{R}$, $\mathbf{w} \in W$, $(\lambda \mathbf{w}) \in W$
  (ii) whenever $\mathbf{w}_1, \mathbf{w}_2 \in W$, $(\mathbf{w}_1 + \mathbf{w}_2) \in W$

(b) (5 pts) a **basis** for a subspace W of a vector space V

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a subspace W if
  - $W = span(\mathbf{v}_1, \ldots, \mathbf{v}_k)$
  - If $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, then coefficients $c_1, \ldots, c_k$ are unique.

(c) (5 pts) a set of **linearly independent** vectors

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set if the only solution of

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

is

$$x_1 = 0, \ldots, x_k = 0$$

(d) (5 pts) a **linear transformation** between two vector spaces V and W.

- A linear transformation between two vector space V and W is a function $T : V \to W$ such that
  (i) $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in V$
  (ii) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

2. Suppose each of the following augmented matrices is a Row Echelon Form of the augmented matrix of a linear system $\mathbf{Ax} = \mathbf{b}$. Describe the original system (i.e., how many equations in how many unknowns) and describe the solution space of the corresponding linear system. (Determine if there are solutions; and, if there are solutions, how many free parameters are needed to describe the general solution.)

(a) (5 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- No solution (There is a pivot in the last column of the third row which implies the contradictory equation $0 = 1$.)

(b) (5 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- There are infinitely many solutions. Since columns 3 and 4 don't have pivots, there will be two free parameters in the general solution.

(c) (5 pts) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- There is a unique solution (since each column of the agumented matrix in RREF has a pivot).

3. (10 pts) Solve the following linear system, expressing the solution set as a hyperplane.

$$\begin{aligned} -x_1 - x_3 + x_4 &= 1 \\ x_1 + x_3 + x_4 &= 1 \\ x_1 + x_4 &= 1 \end{aligned}$$

4. (10 pts) Compute the inverse of $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

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$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|cc} 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{array}\right.$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] = \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$$

and so

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

5. (10 pts) Let $W = \{[x, y] \in \mathbb{R}^2 \mid x + y = 3 \in \mathbb{R}\}$. Prove or disprove that W is a subspace of $\mathbb{R}^2$.

- Every vector in W is of the form $[x, -x]$, $x \in \mathbb{R}$. If $\mathbf{w} = [-x, x] \in W$, we have

$$\lambda \in \mathbb{R},\ \mathbf{w} = [x, -x] \in W \quad \Rightarrow \quad (\lambda \mathbf{w}) = [\lambda x, -\lambda x] \in W$$

so W is closed under scalar multiplication.
If $\mathbf{w}_1 = [x, -x]$, $\mathbf{w}_2 = [y, -y] \in W$, then

$$(\mathbf{w}_1 + \mathbf{w}_2) = [x + y, -(x + y)] \in W$$

and so W is also closed under vector addition.
Since W is closed under both scalar multiplication and vector addition, W is a subspace.

6. Consider the vectors $\{[1, -1, -1, 1], [2, -1, -2, 0], [1, 0, -1, -1], [3, -2, -3, 1]\} \in \mathbb{R}^4$
(a) (10 pts) Determine if these vectors are linearly independent.

- We form a 4×4 matrix $\mathbf{A}$ using the given vectors as rows.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & -1 & -2 & 0 \\ 1 & 0 & -1 & -1 \\ 3 & -2 & -3 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the REF of $\mathbf{A}$ has two zero rows, the original set of vectors are not linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors (i.e. the subspace spanned by these vectors)?

- The two non-zero rows of the REF of $\mathbf{A}$ provide a basis for the $RowSp(\mathbf{A}) = span([1,-1,-1,1],[2,-1,-2,0],[1,0,-1,-1],[3,-2,-3,1])$.

  Since we have two basis vectors, the dimension of the subspace is 2.

7. Consider the following linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_2 - x_1 \, , \, x_1 - x_2]$.
(a) (10 pts) Find a matrix that represents T.

- $$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(b) (10 pts) Find a basis for the kernel of T (i.e. the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$).

- $$\begin{aligned} Ker(T) &= NullSp(\mathbf{A}_T) = NullSp\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\right) \\ &= NullSp(RREF(\mathbf{A}_T)) = NullSp\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= \left\{\mathbf{x} = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 - x_2 = 0\right\} \\ &= \left\{\begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R}\right\} \\ &= span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \end{aligned}$$

  and so
  $$\text{basis for } Ker(T) = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

8. (a) (15 pts) Find the eigenvalues and the eigenvectors of the following matrix : $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

- We have
  $$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2(-\lambda)$$

  So the eigenvalues of $\mathbf{A}$ are 2 and 0. Since we have two factors of $(2-\lambda)$ in $p_{\mathbf{A}}(\lambda)$ the eigenvalue $\lambda = 2$ has algebraic multiplicity 2. Since we have only one factor of $(0-\lambda)$ in $p_{\mathbf{A}}(\lambda)$, the eigenvalue $\lambda = 0$ has algebraic multiplicity 1.

  2-eigenspace
  $$E_2 = NullSp(\mathbf{A} - (2)\mathbf{I}) = NullSp\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
  $$= span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

  Since the 2-eigenspace of $\mathbf{A}$ has only one basis vector, the geometric multiplicity of the eigenvalue 2 is 1.

0-eigenspace

$$E_0 = NullSp\left(\mathbf{A} - (0)\,\mathbf{I}\right) = NullSp\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= span\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right)$$

Since the 0-eigenspace of **A** has only one basis vector, the geometric multiplicity of the eigenvalue 0 is 1.

(b) (5 pts) What are the algebraic multiplicities and geometric multiplicities of the eigenvalues of **A**?

- We have

eigenvalue	basis for eigenspace	alg. mult.	geom. mult.
2	$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$	2	1
0	$\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right\}$	1	1

(c) (5 pts) Is this matrix diagonalizable?

- No. We need 3 linearly independent eigenvectors in order to diagonalize a 3×3 matrix. But we only found two linearly independent eigenvectors; and so **A** is not diagonalizable.

9. (10 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

- First, we need to find the eigenvalues and eigenvectors of $\mathbf{A}$.

$$p_{\mathbf{A}}(\lambda) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4)$$

So the eigenvalues of $\mathbf{A}$ are 2 and 4. Next, we look for the corresponding eigenvectors:

2-eigenspace

$$E_2 = NullSp\left(\begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$
$$= span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

4-eigenspace

$$E_4 = NullSp\left(\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right)$$
$$= span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We use the eigenvalues of $\mathbf{A}$ to construct the diagonal matrix $\mathbf{D}$ and then use the corresponding eigenvectors as the columns of the matrix $\mathbf{C}$. Thus,

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

10. (15 pts)Let $\mathbf{v} = [2,1,0]$ and let $W = span\left([1\ ,1,0]\ ,\ [1,0,1]\right)$. Find the orthogonal decomposition $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$ of $\mathbf{v}$ with respect to the subspace W.

- Since $\{[1\ ,1,0]\ ,\ [1,0,1]\}$ are linearly independent, they'll provide a basis for W.
- Next, we need a basis for $W^\perp = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$

$$W^\perp = NullSp\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right)$$
$$= span\left([-1,1,1]\right)$$

- Combining the basis for $W^\perp$ with the basis for W, we obtain a basis $B = \{[1\ ,1,0]\ ,\ [1,0,1]\ ,[-1,1,1]$ for $\mathbb{R}^3$
- Next, we find the coordinate vector $\mathbf{v}_B$ of $\mathbf{v}$ with respect to the basis B. This means solving the linear system

$$x_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

This linear system has the following augmented matrix

$$[\mathbf{A} \mid \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

which row reduces to

$$[\mathbf{I} \mid \mathbf{v}_B] = \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

And so

$$x_1 = \frac{4}{3} \quad , \quad x_2 = \frac{1}{3} \quad , \quad x_3 = -\frac{1}{3}$$

Thus,

$$\mathbf{v} = \left(\frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The first two terms live in W and their sum is $\mathbf{v}_W$, the last term lives in $W^\perp$ and it corresponds to $\mathbf{v}_\perp$. Thus,

$$\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$$

with

$$\mathbf{v}_W = \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\mathbf{v}_\perp = -\frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} :$$