Math 3013.25578
Solutions to Final Exam
May 2, 2022

1. Give the definitions of the following linear algebraic concepts:

(a) (5 pts) a **subspace** of a vector space V.

- A subspace of vector space V is a subset W of V such that
  (i) whenever $\lambda \in \mathbb{R}$, $\mathbf{w} \in W$, $\quad (\lambda \mathbf{w}) \in W$
  (ii) whenever $\mathbf{w}_1, \mathbf{w}_2 \in W$, $\quad (\mathbf{w}_1 + \mathbf{w}_2) \in W$

(b) (5 pts) a **basis** for a subspace W of a vector space V

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a subspace W if
  - $W = span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k)$
  - If $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, then coefficients $c_1, \ldots, c_k$ are unique.

(c) (5 pts) a set of **linearly independent** vectors

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set if the only solution of
$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$
is
$$x_1 = 0, \ldots, x_k = 0$$

(d) (5 pts) a **linear transformation** between two vector spaces V and W.

- A linear transformation between two vector space V and W is a function $T : V \to W$ such that
  (i) $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in V$
  (ii) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

2. Suppose each of the following augmented matrices is a Row Echelon Form of the augmented matrix of a linear system $\mathbf{Ax} = \mathbf{b}$. Describe the original system (i.e., how many equations in how many unknowns) and describe the solution space of the corresponding linear system. (Determine if there are solutions; and, if there are solutions, how many free parameters are needed to describe the general solution.)

(a) (5 pts) $\left[\begin{array}{cccccc} 0 & -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$: 4 equations in 5 unknowns. Solutions with 2 free parameters (x_1 and x_5).

(b) (5 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$: 4 equations in 4 unknowns. There is no solution since the 3rd row implies a contradictory equation.

(c) (5 pts) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$: 4 equations in 3 unknowns. There is a unique solution (since each column of the coefficient part of the augmented matrix in RREF has a pivot).

3. (10 pts) Solve the following linear system, expressing the solution set as a hyperplane.

$$\begin{aligned} x_1 - x_3 + x_4 &= 1 \\ x_1 + x_3 + x_4 &= 1 \\ x_1 + x_4 &= 1 \end{aligned}$$

- We begin by writing down the augmented matrix for this linear system and row reducing it to Reduced Row Echelon Form.

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2 \to \frac{1}{2}R_2 \\ R_3 \to R_3 - R_2 \\ R_1 \to R_1 + R_2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Converting back to equations (and noting that x_2 and x_4 will be the free parameters in the solution)

$$\left.\begin{array}{r} x_1 + x_4 = 1 \\ x_3 = 0 \\ 0 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = 1 - x_4 \\ x_3 = 0 \end{array}\right.$$

and so a solution vector must be of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

4. (10 pts) Compute the inverse of $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

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$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] = \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$$
and so
$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

5. (10 pts) Let $W = \{[x, y] \in \mathbb{R}^2 \mid x + 2y = 3\}$. Prove or disprove that W is a subspace of $\mathbb{R}^2$.

- Consider the vector $\mathbf{w} = [1, 1]$. $\mathbf{w}$ belongs to W since $(1) + 2(1) = 3$. If we scalar multiply $\mathbf{w}$ by the number 0, we get
$$\lambda\mathbf{w} = [0, 0]$$
which does not lie in W since $(0) + 2(0) = 0 \neq 3$. This means W is not closed under scalar multiplication and so W **is not a subspace.**

6. Consider the vectors $\{[1,1,1,1],[1,2,-1,1],[2,3,0,2],[3,4,1,3]\} \in \mathbb{R}^4$

(a) (10 pts) Determine if these vectors are linearly independent.

- We form a 4×4 matrix $\mathbf{A}$ using the given vectors as rows.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 3 \\ 3 & 4 & 1 & 3 \end{bmatrix} \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the REF of $\mathbf{A}$ a zero row, the original set of vectors are not linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors (i.e. the subspace spanned by these vectors)?

- The two non-zero rows of the REF of $\mathbf{A}$ provide a basis for $RowSp\,(\mathbf{A}) = span\,([1,1,1,1],[1,2,-1,1]$ Since we have two basis vectors, the dimension of the subspace is 2.

7. Consider the following linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T\,([x_1,x_2,x_3]) = [x_2 - x_1\,,\, x_1 - x_2]$.

(a) (10 pts) Find a matrix that represents T.

- $$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T\,([1,0,0]) & T\,([0,1,0]) & T\,([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(b) (10 pts) Find a basis for the kernel of T (i.e. the set of vectors $\mathbf{x}$ such that $T\,(\mathbf{x}) = \mathbf{0}$).

- $$\begin{aligned} Ker\,(T) &= NullSp\,(\mathbf{A}_T) = NullSp\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\right) \\ &= NullSp\,(RREF\,(\mathbf{A}_T)) = NullSp\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= \left\{\mathbf{x} = [x_1,x_2,x_3] \in \mathbb{R}^3 \mid x_1 - x_2 = 0\right\} \\ &= \left\{\begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R}\right\} \\ &= span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \implies \text{basis for } Ker\,(T) = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \end{aligned}$$

8. (a) (15 pts) Find the eigenvalues and the eigenvectors of the following matrix : $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

- We have

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2(-\lambda)$$

So the eigenvalues of $\mathbf{A}$ are 2 and 0. Since we have two factors of $(2-\lambda)$ in $p_{\mathbf{A}}(\lambda)$ the eigenvalue $\lambda = 2$ has algebraic multiplicity 2. Since we have only one factor of $(0-\lambda)$ in $p_{\mathbf{A}}(\lambda)$, the eigenvalue $\lambda = 0$ has algebraic multiplicity 1.

2-eigenspace

$$E_2 = NullSp(\mathbf{A} - (2)\mathbf{I}) = NullSp\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Since the 2-eigenspace of $\mathbf{A}$ has only one basis vector, the geometric multiplicity of the eigenvalue 2 is 1.

0-eigenspace

$$E_0 = NullSp(\mathbf{A} - (0)\mathbf{I}) = NullSp\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= span\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right)$$

Since the 0-eigenspace of $\mathbf{A}$ has only one basis vector, the geometric multiplicity of the eigenvalue 0 is 1.

(b) (5 pts) What are the algebraic multiplicities and geometric multiplicities of the eigenvalues of $\mathbf{A}$?

- We have

eigenvalue	basis for eigenspace	alg. mult.	geom. mult.
2	$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$	2	1
0	$\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right\}$	1	1

(c) (5 pts) Is this matrix diagonalizable?

- No. We need 3 linearly independent eigenvectors in order to diagonalize a 3×3 matrix. But we only found two linearly independent eigenvectors; and so $\mathbf{A}$ is not diagonalizable.

9. (10 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

- First, we need to find the eigenvalues and eigenvectors of $\mathbf{A}$.

  $$p_{\mathbf{A}}(\lambda) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

  So the eigenvalues of $\mathbf{A}$ are 2 and 4. Next, we look for the corresponding eigenvectors:

  2-eigenspace

  $$\begin{aligned} E_2 &= NullSp\left(\begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

  4-eigenspace

  $$\begin{aligned} E_4 &= NullSp\left(\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

  We use the eigenvalues of $\mathbf{A}$ to construct the diagonal matrix $\mathbf{D}$ and then use the corresponding eigenvectors as the columns of the matrix $\mathbf{C}$. Thus,

  $$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

10. Let $W = span\left([1\ ,1,1]\ ,\ [0,1,1]\right)$.

(a) (7 pts) Find the orthogonal complement $W_\perp = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$.

- Note that the generators of W are clearly linearly independent, and so basis vectors for W. To lie in $W_\perp$, a vector $\mathbf{x}$ must be perpendicular to each basis vector of W. Thus, setting $\mathbf{b}_1 = [1,1,1]$, $\mathbf{b}_2 = [0,1,1]$

$$\mathbf{x} \in W_\perp \quad \Rightarrow \quad \left\{ \begin{array}{l} \mathbf{b}_1 \cdot \mathbf{x} = 0 \\ \mathbf{b}_2 \cdot \mathbf{x} = 0 \end{array} \right. \quad \Rightarrow \quad \left[\begin{array}{ccc} \leftarrow & \mathbf{b}_1 & \rightarrow \\ \leftarrow & \mathbf{b}_2 & \rightarrow \end{array} \right] \mathbf{x} = \mathbf{0}$$

Thus,

$$\begin{aligned} W_\perp &= NullSp\left(\begin{array}{ccc} \leftarrow & \mathbf{b}_1 & \rightarrow \\ \leftarrow & \mathbf{b}_2 & \rightarrow \end{array} \right) = NullSp\left(\left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right] \right) = NullSp\left(\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array} \right] \right) \\ &= \left\{ \mathbf{x} \in \mathbb{R}^3 \mid x_1 = 0\ ,\ x_2 = -x_3 \right\} = span\left(\left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array} \right] \right) \end{aligned}$$

(b) (8 pts) Let $\mathbf{v} = [1,-2,0]$. Find the orthogonal decomposition $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$; $\mathbf{v}_W \in W$, $\mathbf{v}_\perp \in W_\perp$, of $\mathbf{v}$ with respect to W.

- $B = \{[1,1,1]\,,[0,1,1]\} \cup \{[0,-1,1]\}$ will be a basis for the entire vector space $\mathbb{R}^3$. Let us find the coordinates of $\mathbf{v}$ with respect to the basis B

$$c_1 \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] + c_2 \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] + c_3 \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array} \right] = \left[\begin{array}{c} 1 \\ -2 \\ 0 \end{array} \right]$$

This linear system has the augmented matrix

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & 1 & 1 & 0 \end{array} \right]$$

which in turn row reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right] = R.R.E.F.\left([\mathbf{A}|\mathbf{b}]\right) \quad \Rightarrow \quad \left\{ \begin{array}{l} c_1 = 1 \\ c_2 = -2 \\ c_3 = 1 \end{array} \right.$$

and so

$$\begin{aligned} \mathbf{v} &= \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] - 2 \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] + \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array} \right] \\ &= \left(\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] - 2 \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] \right) + \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array} \right] \\ &= \mathbf{v}_W + \mathbf{v}_\perp \end{aligned}$$

where

$$\mathbf{v}_W = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \in W$$

$$\mathbf{v}_\perp = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \in W_\perp$$