Math 3013.62663
Final Exam
December 10, 2021 : 10:00am - 11:50am

Name:______________________________

1. Complete the following definitions

(a) (5 pts) A **subspace** of a vector space V is ...

(b) (5 pts) A **basis** for a subspace W of a vector space V is ...

(c) (5 pts) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a **linearly independent** set of vectors if ...

(d) (5 pts) A **linear transformation** between two vector spaces V and W is ...

2. For each of the following augmented matrices, describe the solution space of the corresponding linear system. (Determine if there are solutions; and, if there are solutions, how many free parameters are needed to describe the general solution.)

(a) (5 pts) $\left[\begin{array}{ccccc|c} 0 & -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(b) (5 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(c) (5 pts) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

3. (10 pts) Compute the inverse of $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

4. (10 pts) Let $W = \{[x, y] \in \mathbb{R}^2 \mid x + 2y = 3\}$. Prove or disprove that W is a subspace of $\mathbb{R}^2$.

5. Consider the vectors $\{[1, 1, 1, 1], [1, 2, -1, 1], [2, 3, 0, 2], [3, 4, 1, 3]\} \in \mathbb{R}^4$

(a) (10 pts) Determine if these vectors are linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors (i.e. the subspace spanned by these vectors)?

6. Consider the following linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2.x_3]) = [x_2 - x_1\ ,\ x_1 - x_2]$.

(a) (10 pts) Find a matrix that represents T.

(b) (10 pts) Find a basis for the kernel of T (i.e. the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$).

7. (a) (10 pts) Find the eigenvalues and the eigenvectors of the following matrix : $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(b) (5 pts) What are the algebraic and geometric multiplicities of each eigenvalue?

(c) (5 pts) Is this matrix diagonalizable?

8. (15 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

9. Let $W = span\,([1\ ,1,1]\,,\ [0,1,1]) \subset \mathbb{R}^3$.

(a) (7 pts) Find the orthogonal complement $W_\perp = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{x} \in W\}$ of W in $\mathbb{R}^3$.

(b) (8 pts) Let $\mathbf{v} = [1, -2, 0]$, find the orthogonal decomposition $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$; $\mathbf{v}_W \in W$, $\mathbf{v}_\perp \in W_\perp$, of $\mathbf{v}$ with respect to W.

10. (10 pts) Find an orthonormal basis for the subspace W generated by the vectors $\mathbf{v}_1 = [1, -1, 1]$ and $\mathbf{v}_2 = [1, 0, 2]$.