Math 3013.62979
Solutions to Final Exam
2:00pm – 4:00pm, December 8, 2020

1. Definitions
2.
   (a) No solution (There is a pivot in the last column of the third row which implies the contradictory equation $0 = 1$.)
   (b) There are infinitely many solutions. Since columns 3 and 4 don't have pivots, there will be two free parameters in the general solution.
   (c) There is a unique solution (since each column of the agumented matrix in RREF has a pivot).
3.

$$\mathbf{A}^{-1} = \left[\left[\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{array}\right]\right.$$

4.
   - Since W is closed under both scalar multiplication and vector addition, W is a subspace.
5. Consider the vectors $\{[1,-1,-1,1],[2,-1,-2,0],[1,0,-1,-1],[3,-2,-3,1]\} \in \mathbb{R}^4$

(a) (10 pts) Determine if these vectors are linearly independent.

- We form a 4×4 matrix $\mathbf{A}$ using the given vectors as rows.

$$\mathbf{A} = \left[\begin{array}{cccc} 1 & -1 & -1 & 1 \\ 2 & -1 & -2 & 0 \\ 1 & 0 & -1 & -1 \\ 3 & -2 & -3 & 1 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cccc} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  Since the REF of $\mathbf{A}$ has two zero rows, the original set of vectors are not linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors (i.e. the subspace spanned by these vectors)?

- The two non-zero rows of the REF of $\mathbf{A}$ provide a basis for the $RowSp(\mathbf{A}) = span([1,-1,-1,1],[2,-1,-2,0],[1,0,-1,-1],[3,-2,-3,1])$.
  Since we have two basis vectors, the dimension of the subspace is 2.

6.

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$$\mathbf{A}_T = \left[\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{array}\right] = \left[\begin{array}{ccc} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

(b) (10 pts) Find a basis for the kernel of T (i.e. the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$).

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$$\text{basis for } Ker(T) = \left\{ \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] \right\}$$

7.

(a) (15 pts) Find the eigenvalues and the eigenvectors of the following matrix : $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(b) (5 pts) What are the algebraic multiplicities and geometric multiplicities of the eigenvalues of $\mathbf{A}$?

- We have

eigenvalue	basis for eigenspace	alg. mult.	geom. mult.
2	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$	2	1
0	$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$	1	1

(c) (5 pts) Is this matrix diagonalizable?

- No. We need 3 linearly independent eigenvectors in order to diagonalize a 3×3 matrix. But we only found two linearly independent eigenvectors; and so $\mathbf{A}$ is not diagonalizable.

8. (10 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

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$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

9. (10 pts) Find an orthonormal basis for the subspace W generated by the vectors $\mathbf{v}_1 = [1, -1, 0]$ and $\mathbf{v}_2 = [1, 0, 1]$

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$$\mathbf{n}_1 = \frac{\mathbf{o}_1}{\sqrt{\mathbf{o}_1 \cdot \mathbf{o}_1}} = \frac{1}{\sqrt{2}}[1, -1, 0] = \left[\frac{1}{\sqrt{2}} \, , \, -\frac{1}{\sqrt{2}} \, , \, 0 \right]$$

$$\mathbf{n}_2 = \frac{\mathbf{o}_2}{\sqrt{\mathbf{o}_2 \cdot \mathbf{o}_2}} = \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + (1)^2}} \left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \end{array} \right] = \left[\frac{1}{\sqrt{6}} \, , \, \frac{1}{\sqrt{6}} \, , \, \frac{2}{\sqrt{6}} \right]$$

10.

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$$[\mathbf{A} \mid \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to [\mathbf{I} \mid \mathbf{v}_B] = \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

and

$$\mathbf{v}_W = \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\mathbf{v}_\perp = -\frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} :$$