Math 3013.69033
Solutions to Final Exam
6:00pm - 8:00pm CDT, December 10, 2020
Name:________________________________

1. Give the definitions of the following linear algebraic concepts:

(a) (5 pts) a **subspace** of a vector space V.

- A subspace of vector space V is a subset W of V such that
  (i) whenever $\lambda \in \mathbb{R}$, $\mathbf{w} \in W$, $\quad (\lambda \mathbf{w}) \in W$
  (ii) whenever $\mathbf{w}_1, \mathbf{w}_2 \in W$, $\quad (\mathbf{w}_1 + \mathbf{w}_2) \in W$

(b) (5 pts) a set of **linearly independent** vectors

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set if the only solution of
$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$
  is
$$x_1 = 0, \ldots, x_k = 0$$

(c) (5 pts) a **linear transformation** between two vector spaces V and W.

- A linear transformation between two vector space V and W is a function $T : V \to W$ such that
  (i) $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in V$
  (ii) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

(d) (5 pts) a **basis** for a subspace W of a vector space V

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a subspace W if
  (i) $W = \mathit{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$
  (ii) If $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, then coefficients $c_1, \ldots, c_k$ are unique.

2. For each of the following augmented matrices, describe the solution space of the corresponding linear system. (Determine if there are solutions; and, if there are solutions, how many free parameters are needed to describe the general solution.)

(a) (5 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- The corresponding linear system has infinitely many solutions. Since columns 3 and 4 do not contain pivots, the solution will have 2 free parameters.

(b) (5 pts) $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- The corresponding linear system has a unique solution (no free parameters)

(c) (5 pts) $\left[\begin{array}{ccccc|c} 0 & -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- The corresponding linear system has no solution (the third row corresponds to the equation $0 = 1$, which is a mathematical contradiction).

3. (10 pts) Compute the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

•

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] = \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$$

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

4. (10 pts) Let $W = \{[x, y] \in \mathbb{R}^2 \mid x + 2y = 3 \in \mathbb{R}\}$. Prove or disprove that W is a subspace of $\mathbb{R}^2$.

- Let $\mathbf{w} = [1, 1]$. Since $(1) + 2(1) = 3$, this vector is in W. Now consider the scalar multiple of $\mathbf{w}$ by $\lambda = 0$. We have

$$(0)\,\mathbf{w} = [0, 0]$$

but

$$(0) + (2)(0) = 0 \neq 3$$

and so for this λ, $\lambda\mathbf{w} \notin W$. This means W is not closed under scalar multiplication, and so W is not a subspace.

5. Consider the following linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2.x_3]) = [x_2 - x_3\ ,\ x_1 - x_3]$.

(a) (10 pts) Find a matrix that represents T.

•

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

(b) (10 pts) Find a basis for the kernel of T (i.e. the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$).

•

$$\begin{aligned} \ker(T) &= NullSp(\mathbf{A}_T) = NullSp\left(\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\right) \\ &= NullSp\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \\ &\Rightarrow \quad \text{basis for } \ker(T) = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \end{aligned}$$

6. Consider the vectors $\{[1,-2,2,1],[1,-1,3,1],[0,1,1,0],[2,-3,5,2]\} \in \mathbb{R}^4$

(a) (10 pts) Determine if these vectors are linearly independent.

- We form a matrix $\mathbf{A}$ using the given vectors as rows and row reduce $\mathbf{A}$ to Row Echelon Form
$$\begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & -3 & 5 & 2 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
since we have a zero row (or 2 zero rows), the original set of vectors are not linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors (i.e. the subspace spanned by these vectors)?

- Since the REF of $\mathbf{A}$ has 2 non-zero rows, its row space is 2-dimensional. Since this row space coincides with the subspace generated by the original list of vectors, this subspace is also 2-dimensional

7. (a) (10 pts) Find the eigenvalues and the eigenvectors of the following matrix : $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

-
$$p_{\mathbf{A}}(\lambda) = \det\left(\begin{bmatrix} 2-\lambda & 2 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2(-\lambda)$$
$$\Rightarrow \quad \text{eigenvalues } = \{2, 0\}$$
2-eigenspace
$$E_{\lambda=2} = NullSp\left(\mathbf{A}-2\mathbf{I}\right) = NullSp\left(\begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = NullSp\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \left\{\begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} \mid x_1, x_3 \in \mathbb{R}\right\} = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$
0-eigenspace
$$E_{\lambda=0} = NullSp\left(\mathbf{A}-(0)\,\mathbf{I}\right) = NullSp\left(\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$
$$= \left\{\begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} \mid x_2 \in \mathbb{R}\right\} = span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

(b) (5 pts) What are the algebraic and geometric multiplicities of each eigenvalue of $\mathbf{A}$.

- The algebraic multiplicity of an eigenvalue r is the number of factors of $(\lambda - r)$ in the characteristic polynomial $p_{\mathbf{A}}(\lambda)$ of $\mathbf{A}$. The geometric multiplicity of an eigenvalue r is the dimension of the corresponding eigenspace (the number of basis vectors). Thus,

(c) (5 pts) Is this matrix diagonalizable?

- We need 3 linearly independent eigenvectors to diagonalize a 3×3 matrix. Since we found three linearly independent eigenvectors (two for $\lambda = 2$ and one for $\lambda = 0$), **A is diagonalizable**

8. (15 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

- We first need to find the eigenvalues and eigenvectors of $\mathbf{A}$.

$$\begin{aligned} p_{\mathbf{A}}(\lambda) &= \det\begin{pmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{pmatrix} = (1-\lambda)(6-\lambda) + 6 = \lambda^2 - 7\lambda + 12 \\ &= (\lambda - 3)(\lambda - 4) \\ &\Rightarrow \quad \text{eigenvalues} = \{3, 4\} \end{aligned}$$

3-eigenspace

$$\begin{aligned} E_{\lambda=3} &= NullSp(\mathbf{A} - (3)\mathbf{I}) = NullSp\left(\begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right) = span\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) \end{aligned}$$

4-eigenspace

$$\begin{aligned} E_{\lambda=4} &= NullSp(\mathbf{A} - (4)\mathbf{I}) = NullSp\left(\begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \end{aligned}$$

The diagonal matrix $\mathbf{D}$ is formed from the eigenvalues of $\mathbf{A}$ and the diagonalizing matrix $\mathbf{C}$ is formed by using the eigenvectors of $\mathbf{A}$ as columns (following your ordering of eigenvalues). Thus,

$$\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

9. (15 pts) Let $\mathbf{v} = [2, 1, 2]$ and let $W = span([0, 1, 1]\ ,\ [1, 1, 0])$. Find the orthogonal decomposition $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$ of $\mathbf{v}$ with respect to the subspace W.

- Note that the two vectors generating W are linearly independent and so form a basis $B_W = \{[0, 1, 1]\ ,\ [1, 1, 0]\}$ for W.
- Next we need a basis for $W^\perp = \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$. We have

$$W^\perp = NullSp\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$B_{W^\perp} = \{[1, -1, 1]\}$$

- $B = B_W \cup B_{W^\perp} = \{[0, 1, 1]\ ,\ [1, 1, 0]\ ,\ [1, -1, 1]\}$ will be a basis for $\mathbb{R}^3$. We now find the coordinates $[c_1, c_2, c_3]$ of $\mathbf{v}$ with respect to B, i.e., we solve

$$c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad c_1 = 1 \quad , \quad c_2 = 1 \quad , \quad c_3 = 1$$

And so

$$\mathbf{v} = (1)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The sum of the first two vector terms on the right is $\mathbf{v}_W$ and the last term is $\mathbf{v}_\perp$. Thus,

$$\mathbf{v}_W = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v}_\perp = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

10. (10 pts) Find an orthonormal basis for the subspace W generated by the vectors $\mathbf{v}_1 = [1,1,1]$ and $\mathbf{v}_2 = [1,0,1]$

- We'll first construct an orthogonal basis $\{\mathbf{o}_1, \mathbf{o}_2\}$ from the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of W. Thus, we take
$$\mathbf{o}_1 = \mathbf{v}_1 = [1,1,1]$$
and then construct a second orthogonal basis vector by removing from $\mathbf{v}_2$ the component that runs parallel to $\mathbf{o}_1$
$$\begin{aligned}\mathbf{o}_2 &= \mathbf{v}_2 - \frac{\mathbf{o}_1 \cdot \mathbf{v}_2}{\mathbf{o}_1 \cdot \mathbf{o}_1}\mathbf{o}_1 = [1,0,1] - \frac{1+0+1}{1+1+1}[1,1,1] \\ &= \left[\begin{array}{ccc} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right]\end{aligned}$$
We now have an orthogonal basis $\{\mathbf{o}_1, \mathbf{o}_2\}$ for W, but it's not an orthonormal basis since the basis vectors do not have length 1. But this is easily remedied by dividing the vectors $\mathbf{o}_i$ by their lengths $\sqrt{\mathbf{o}_i \cdot \mathbf{o}_i}$. Thus,
$$\begin{aligned}\mathbf{n}_1 &= \frac{\mathbf{o}_1}{\sqrt{\mathbf{o}_1 \cdot \mathbf{o}_1}} = \frac{1}{\sqrt{3}}[1,1,1] = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \\ \mathbf{n}_2 &= \frac{\mathbf{o}_2}{\sqrt{\mathbf{o}_2 \cdot \mathbf{o}_2}} = \frac{3}{\sqrt{6}}\left[\begin{array}{ccc} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] = \left[\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]\end{aligned}$$
And so our orthonormal basis is
$$\{\mathbf{n}_1, \mathbf{n}_2\} = \left\{\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] , \left[\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]\right\}$$