LECTURE 1

Vectors and Vector Spaces

1. Vectors

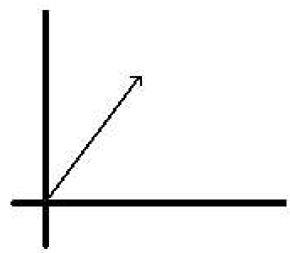
There are three fundamental ways of thinking about n-dimensional vectors:

: **Algebraically**; as ordered sets of n real numbers. For example, if $v_1, v_2, \ldots, v_n$ is a sequence of n numbers, we denote by

$$\mathbf{V} = (v_1, v_2, \ldots, v_n)$$

the corresponding (algebraic) vector. We shall refer to the v_i as the *components* of the vector $\mathbf{V}$.

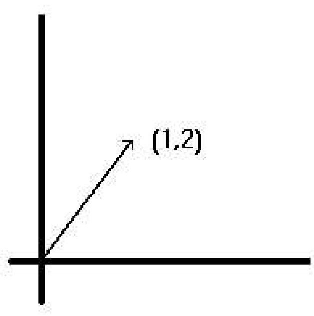
: **Geometrically**; as directed line segments (with its tail at the origin) in an n-dimensional space. In other words, a vector is essentially a figure like



We shall denote the length of a vector $\mathbf{V}$ by $|\mathbf{V}|$.

: **Physically**; as quantities with both a magnitude and direction. For example, the position of an object with respect to a fixed origin is a quantity with both a magnitude (the distance of the object to the origin) and a direction.

The first two (mathematical) points of view are of course completely equivalent and it is trivial (yet often helpful) to pass back and forth between the algebraic and geometric points of view. For example, in two dimensions the "geometric vector" that corresponds to the "algebraic vector" $(1, 2)$ is the directed line segment in the plane that has its tail at the origin and its head at the point $(1, 2)$. In general, the coordinates of the head of a "geometric vector" correspond precisely to the ordered set of numbers that comprise the corresponding "algebraic vector".



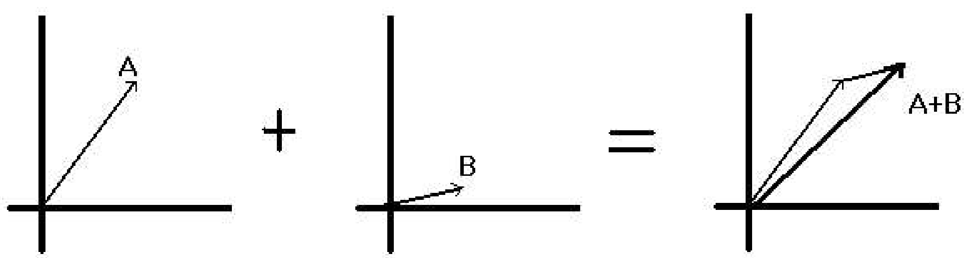
2. Vector Operations

We shall denote by $\mathbb{R}^n$ the space of n-dimensional vectors. This notation arises from the observation that the specification of a 1-dimensional vector requires 1 real number (and $\mathbb{R}^1 = \mathbb{R}$ is standard notation for the set of real numbers)

2.1. Vector Addition: In the algebraic interpretation of vectors, the sum of two vectors $\mathbf{A} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{B} = (b_1, b_2, \ldots, b_n)$ is the vector (i.e. ordered set of numbers) whose components are the sum of the corresponding components of $\mathbf{A}$ and $\mathbf{B}$;

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

Geometrically, the sum of two vectors $\mathbf{A}$ and $\mathbf{B}$ is the vector obtained by "parallel-transporting" the tail of $\mathbf{B}$ to the tip of $\mathbf{A}$ and then drawing a directed line segment from the origin to the (new) position of the tip of $\mathbf{B}$:



Definition 1.1. *The n-dimensional* ***null vector*** $\mathbf{0}$ *is the vector for which each of its n components is 0.*

Theorem 1.2. *(Properties of Vector Addition.) Let* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$. *Then*

(1) *Vector addition is associative:* $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
(2) *Vector addition is commutative:* $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
(3) $\mathbf{0}$ *is an additive identity:* $\mathbf{0} + \mathbf{w} = \mathbf{w}$

2.2. Scalar Multiplication. In the algebraic representation of vectors scalar multiplication by a number λ is the operation corresponding to multiplying each of the components of a vector $\mathbf{V} = (v_1, v_2, \ldots, v_n)$ by the number λ

$$\lambda \mathbf{V} = (\lambda v_1, \lambda v_2, \ldots, \lambda v_n)$$

In the geometric representation, the vector $\lambda \mathbf{V}$ is, so long as λ is non-negative, the vector with the same direction as $\mathbf{V}$ but whose length has been rescaled by a factor of λ; if λ is negative, then $\lambda \mathbf{V}$ is the vector whose direction is exactly the opposite of $\mathbf{V}$ and whose length has been rescaled by a factor of $|\lambda|$.

Theorem 1.3. *(Properties of Scalar Multiplication.) Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and let* r, s *be any scalars in* $\mathbb{R}$. *Then*

(1) $-\mathbf{v} \equiv (-1)\mathbf{v}$ *is the additive inverse of* $\mathbf{v}$*:* $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
(2) *Scalar multiplication is distributive with respect to vector addition:* $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$
(3) *Scalar multiplication is distributive with respect to scalar addition:* $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$
(4) *Scalar mulitiplication is associative:* $r(s\mathbf{v}) = (rs)\mathbf{v}$
(5) *1 is a mulitiplicative identity for scalar multiplication:* $(1)\mathbf{v} = \mathbf{v}$

2.3. Summary: Fundamental Properties of the Vector Space $\mathbb{R}^n$. The vector space $\mathbb{R}^n$ consists of all ordered lists of n real numbers

$$\mathbb{R}^n \equiv \{[a_1, \ldots, a_n] \mid a_1, \ldots, a_n \in \mathbb{R}\}$$

On this set, two operations are defined

$$\begin{aligned} \text{vector addition} &: \quad [a_1, \ldots, a_n] + [b_1, \ldots, b_n] = [a_1 + b_1, \ldots, a_n + b_n] \\ \text{scalar multiplication} &: \quad \lambda [a_1, \ldots, a_n] \equiv [\lambda a_1, \ldots, \lambda a_n] \end{aligned}$$

and the following 8 identities hold

DEFINITION 1.4.
- (1) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ *(associativity of vector addition)*
(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *(commutativity of vector addition)*
(3) *There exists an element* $\mathbf{0} \in V$ *such that* $\mathbf{v} + \mathbf{0} = \mathbf{v}$ *for all* $\mathbf{v} \in V$. *(additive identity.)*
(4) *For each* $\mathbf{v} \in V$ *there exists an element* $-\mathbf{v} \in V$ *such that* $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. *(additive inverses)*
(5) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ *(scalar multiplication is distributive with respect to vector addition).*
(6) $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$. *(scalar multiplication is distributive with respect to addition of scalars)*
(7) $r(s\mathbf{v}) = (rs)\mathbf{v}$ *(scalar multiplication preserves associativity of multiplication in* $\mathbb{R}$.*)*
(8) $(1)\mathbf{v} = \mathbf{v}$ *(preservation of scale).*

2.4. More General Vector Spaces. Each of the 8 properties of vectors in $\mathbb{R}^n$ is easy to verify by direct calculation; but the real point of this formality is circumstance that there are many other examples of sets with similar properties. In fact, the reason you're studying linear algebra and $\mathbb{R}^n$ in particular is because the calculational tools for $\mathbb{R}^n$ provide the means for doing calculations on more general vector spaces. Let me give a couple of examples of more other vector spaces that can be modelled by vectors in $\mathbb{R}^n$.

2.4.1. *Function Spaces.* Let V be the set of all functions on the real line $\mathbb{R}$. On this set we can define notions of vector addition and scalar multiplication as

$$\begin{aligned} (f + g)(x) &\equiv f(x) + g(x) \\ (\lambda f)(x) &= \lambda f(x) \end{aligned}$$

and if we set $\mathbf{0}_V$ as the function $\mathbf{0}_V(x) = 0$ for all $x \in \mathbb{R}$, it is easy to verify that analogs of 8 properties of vectors in $\mathbb{R}^n$ are satisfied. Because of this connection, linear algebraic concepts are extremely useful in seemingly unrelated mathematical topics like the study of differential equations.

2.4.2. *Vibrations.* For this example, imagine you are in a concert hall listening to a string quartet. Each instrument contributes its own vibrations to the overall sound you hear. In fact, the way in which the notes from different instruments combine to produce the overall sound is a type of vector addition.

Changing the volume of a note or sound is akin to scalar multiplication by the corresponding scaling factor. Silence would correspond to the zero vector in this vector space of sounds and notes.

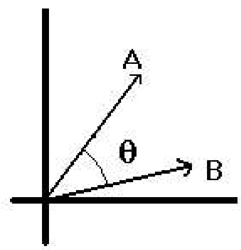
What's left to explain is what the "additive inverse" of a sound or note would be; i.e., if v is a sound, what is $-v$? Well, the idea here is actually the key property of vibrations that is used by "noise-cancelling" headphones. $-v$ would be the "phase reversal" of v. More explicitly, if $p_v(x,t)$ is the air pressure (relative to equilibrium presssure) at position x and time t due to the sound v, then $-v$ would be the sound where the air pressure at position x and time t is $-p_v(x,t)$. Naturally, the combination of the sound v and its phase-reversal $-v$ is a sound corresponding to zero pressure $(p_v(x,t) + (-p_v(x,t)) = 0)$; i.e. silence (the "zero sound").

3. Inner Product of Vectors

The inner product of two algebraic vectors $\mathbf{A} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{B} = (b_1, b_2, \ldots, b_n)$ is the number corresponding to the sum of the products of the corresponding components:

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

In the geometric representation of vectors, the inner product $\mathbf{A} \cdot \mathbf{B}$ is the number obtained by multiplying the product of the lengths of $\mathbf{A}$ and $\mathbf{B}$ by the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$:



$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos(\theta)$$

Note that

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|\,|\mathbf{A}| \cos(0) = |\mathbf{A}|^2$$

or

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{(a_1)^2 + (a_2)^2 + \cdots + (a_n)^2}$$

which can be understood as the generalization of Pythagoras' theorem to n-dimesions

THEOREM 1.5. *For any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and any* $\alpha, \beta \in \mathbb{R}$

(1) $(\alpha \mathbf{u} + \beta \mathbf{v}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) + \beta(\mathbf{v} \cdot \mathbf{w})$
(2) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
(3) $\mathbf{u} \cdot \mathbf{u} \geq 0$
(4) $\mathbf{u} \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u} = \mathbf{0}$, *where* $\mathbf{0} \equiv (0, 0, 0)$

3.1. Cross Product.

For 3-dimensional vectors, and only 3-dimensional vectors, we have also a way of multiplying two vectors to get another vector. In terms of the algebraic representation the cross product $\mathbf{A} \times \mathbf{B}$ of two vectors $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$ is the vector

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \\ &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \end{aligned}$$

In the geometric representation, the cross product of two 3-dimensional vectors $\mathbf{A}$ and $\mathbf{B}$ is the vector whose magnitude is

$$|\mathbf{A}|\,|\mathbf{B}|\,|\sin(\theta)|$$

where θ is the (shortest) angle from $\mathbf{A}$ to $\mathbf{B}$ and whose direction is perpendicular to the plane containing both $\mathbf{A}$ and $\mathbf{B}$ and such that when one faces the plane containing the $\mathbf{A}$ and $\mathbf{B}$ in the direction $\mathbf{A} \times \mathbf{B}$, $\mathbf{B}$ is oriented clockwise from $\mathbf{A}$. This awkward description of the direction of the $\mathbf{A} \times \mathbf{B}$ is stated more simply in terms of the "right hand rule":

> If you point the index finger of your right hand in the direction of $\mathbf{A}$ and the middle finger of your right hand in the direction of $\mathbf{B}$ then the direction of $\mathbf{A} \times \mathbf{B}$ will be the direction of your right thumb.(when it's oriented perpedicularly to the first two fingers).

4. Standard Vectors

One of the most common settings for vectors is the case of vectors in a three dimensional space. Since we commonly label the variables representing the coordinates of a 3-dimensional space by x, y, and z we shall often label the first, second, and components of a 3-dimensional vector $\mathbf{V}$ by V_x, V_y, and V_z. In the set of all 3-dimensional vectors, there are three most fundamental; the unit vectors along the coordinate axes. We shall label these as

$$\mathbf{i} = (1, 0, 0)$$
$$\mathbf{j} = (0, 1, 0)$$
$$\mathbf{k} = (0, 0, 1)$$

These vectors have the property that if $\mathbf{V}$ is *any* vector, its first (or x) component is precisely $\mathbf{V} \cdot \mathbf{i}$, its second (or y) component is precisely $\mathbf{V} \cdot \mathbf{j}$, and its third (or z) component is precisely $\mathbf{V} \cdot \mathbf{k}$. Thus,

$$\begin{aligned} V_x &= \mathbf{V} \cdot \mathbf{i} \quad , \quad V_y = \mathbf{V} \cdot \mathbf{j} \quad , \quad V_z = \mathbf{V} \cdot \mathbf{k} \\ \mathbf{V} &= V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k} \end{aligned}$$

In the more general case (i.e. for vectors in an n-dimensional space), it is more palatable to use numerical indices to label the components of vectors. Thus, the components of an n-dimensional vector $\mathbf{V}$ would be labeled $V_1, V_2, \ldots, V_n$. Also in the general case, unit vectors are often utilized. Thus, we set $\mathbf{e}_i$, the unit vector in the i^{th} direction, to be the vector whose only non-zero component is in the i^{th} slot, and which has a 1 in the i^{th} slot. We then have

$$\begin{aligned} V_i &= \mathbf{V} \cdot \mathbf{e}_i \\ \mathbf{V} &= \sum_{i=1}^{n} V_i\mathbf{e}_i \end{aligned}$$