LECTURE 9

Construction of Bases, Linear Independence and Dimension

A subspace W (for example, the solution set of a set of homogeneous linear equations) can be generated by taking linear combinations of a set of vectors $\{\mathbf{w}_1,\dots,\mathbf{w}_k\}$. The purpose of this lecture is address the question: given a fixed subspace W, how do we know when we've picked enough vectors $\mathbf{w}_1,\dots,\mathbf{w}_k\in W$ so that we can represent **every** other vector in W uniquely in terms of a particular linear combination of the $\mathbf{w}_i$? In the language of Lecture 7, how do we know we have a **basis** for W?

1. Constructing a Basis for a Span of Vectors

Let $\mathbf{w}_1,\dots,\mathbf{w}_k$ be vectors in $\mathbb{R}^n$, and let

$$W \equiv span\left(\mathbf{w}_1,\dots,\mathbf{w}_k\right) \equiv \{\mathbf{w}\in\mathbb{R}^n \mid \mathbf{w}=c_1\mathbf{w}_1+c_2\mathbf{w}_2+\cdots+c_k\mathbf{w}_k \quad ; \quad c_1,\dots,c_k\in\mathbb{R}\} \tag{9.1}$$

Suppose $\{\mathbf{w}_1,\dots,\mathbf{w}_k\}$ is not a basis for W, then by Theorem 8.14 (Lecture 8), we know that we must have a non-trivial solution of

$$\mathbf{0}=r_1\mathbf{w}_1+r_2\mathbf{w}_2+\cdots+r_k\mathbf{w}_k \tag{9.2}$$

that is, a solution for which at least one of the r_i does not equal zero. Without loss of generality (e.g. by reordering the vectors $\mathbf{w}_i$) we can assume it is the last coefficient r_k that does not vanish. Then we can use (9.2) to express $\mathbf{w}_k$ in terms of the vectors $\mathbf{w}_1,\dots,\mathbf{w}_{k-1}$

$$\mathbf{w}_k=-\frac{1}{r_k}\left(r_1\mathbf{w}_1+r_2\mathbf{w}_2+\cdots+r_{k-1}\mathbf{w}_{k-1}\right)$$

It is then easy to see that the smaller set of vectors $\{\mathbf{w}_1,\dots,\mathbf{w}_{k-1}\}$ also generate W: for $\mathbf{w}\in W$ implies

$$\begin{aligned}
\mathbf{w} &= c_1\mathbf{w}_1+\cdots+c_{k-1}\mathbf{w}_{k-1}+c_k\mathbf{w}_k\\
&= c_1\mathbf{w}_1+\cdots+c_{k-1}\mathbf{w}_{k-1}-\frac{c_k}{r_k}\left(r_1\mathbf{w}_1+r_2\mathbf{w}_2+\cdots+r_{k-1}\mathbf{w}_{k-1}\right)\\
&= \left(c_1-\frac{c_kr_1}{r_k}\right)\mathbf{w}_1+\left(c_2-\frac{c_kr_2}{r_k}\right)\mathbf{w}_2+\cdots+\left(c_{k-1}-\frac{c_kr_{k-1}}{r_k}\right)\mathbf{w}_{k-1}\\
&\in span\left(\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_{k-1}\right)
\end{aligned}$$

In other words, if $\{\mathbf{w}_1,\dots,\mathbf{w}_k\}$ is not a basis, we can always find a smaller subset of vectors that generate same subspace. The converse to this statement is also true: if we can not find a smaller (i.e., proper) subset of vectors that generate the subspace $W=span\left(\mathbf{w}_1,\dots,\mathbf{w}_k\right)$, then the vectors $\mathbf{w}_1,\dots,\mathbf{w}_k$ form a basis for W.

EXAMPLE 9.1. Find a basis for $W=span\left(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\right)\subset\mathbb{R}^2$ where

$$\mathbf{w}_1=\begin{bmatrix}1\\2\end{bmatrix}\quad,\quad\mathbf{w}_2=\begin{bmatrix}1\\1\end{bmatrix}\quad,\quad\mathbf{w}_3=\begin{bmatrix}-2\\-1\end{bmatrix}$$

- First we look for nontrivial solutions of

$$r_1\mathbf{w}_1+r_2\mathbf{w}_2+r_3\mathbf{w}_3=r_1\begin{bmatrix}1\\2\end{bmatrix}+r_2\begin{bmatrix}1\\1\end{bmatrix}+r_3\begin{bmatrix}-2\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=\mathbf{0} \tag{9.3}$$

This vector equation is equivalent to the following linear system

$$\begin{aligned} r_1 + r_2 - 2r_3 &= 0 \\ 2r_1 + r_2 - r_3 &= 0 \end{aligned}$$

or the following augmented matrices

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

or

$$\left.\begin{array}{r} r_1 + r_3 = 0 \\ r_2 - 3r_3 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} r_1 = -r_3 \\ r_2 = 3r_3 \end{array}\right. \qquad \text{for some } r_3 \in \mathbb{R}$$

Taking $r_3 = 1$ we thus have a solution with $r_1 = -1$ and $r_2 = 3$. Indeed,

$$(-1)\left[\begin{array}{c} 1 \\ 2 \end{array}\right] + 3\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -2 \\ -1 \end{array}\right] = \left[\begin{array}{c} -1+3=2 \\ -2+3-1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

So

$$-\mathbf{w}_1 + 3\mathbf{w}_2 + \mathbf{w}_3 = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{w}_3 = \mathbf{w}_1 - 3\mathbf{w}_2$$

Because we can express $\mathbf{w}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$

$$W \equiv \mathit{span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \mathit{span}(\mathbf{w}_1, \mathbf{w}_2)$$

and perhaps $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W.

To see if $\{\mathbf{w}_1, \mathbf{w}_2\}$ is indeed a basis, we repeat the calculation above. We first look for non-trivial solutions of

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 = \mathbf{0} \tag{9.4}$$

or

$$\begin{aligned} r_1 + r_2 &= 0 \\ 2r_1 + r_2 &= 0 \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

which corresponds to a linear system with only one solution

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \end{aligned}$$

Since we can't find non-trivial solutions of (9.4), we conclude that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\mathit{span}(\mathbf{w}_1, \mathbf{w}_2) = \mathit{span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \equiv W$.

The following definition formalizes the ideas behind this construction of bases.

DEFINITION 9.2. *Let* $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ *be a set of vectors in* $\mathbb{R}^n$. *A* ***dependence relation*** *among the* $\mathbf{w}_i$ *is an equation of the form*

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k = \mathbf{0} \quad , \quad \textit{with at least one } r_i \neq 0.$$

If such a dependence relation exists, the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ *is a* ***linearly dependent set of vectors****. If such a dependence relation does not exist,then the vectors* $\mathbf{w}_1, \ldots, \mathbf{w}_k$ *are said to be* ***linearly independent.***

2. Dimensions of Subspaces

LEMMA 9.3. *Suppose S is a subspace generated by k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ (in other words, $S = span\left(\mathbf{v}_1, \ldots, v_k\right)$). Then any set $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ of ℓ vectors in S, wth $\ell > k$, is linearly independent.*

Proof. This is proved by induction. I'll show here only the first step of the proof - this at least shows you the basic idea behind the Lemma. Suppose $k = 1$. Then S would be of the form

$$S = span\{\mathbf{v}_1\} = \{\lambda \mathbf{v}_1 \mid \lambda \in \mathbb{R}\}$$

Now suppose $\mathbf{w}_1, \mathbf{w}_2 \in S$. Then for some choice of numbers λ, ρ we have

$$\mathbf{w}_1 = \lambda \mathbf{v}_1 \quad \text{and} \quad \mathbf{w}_2 = \rho \mathbf{v}_2$$

To show that $\{\mathbf{w}_1 \mathbf{w}_2\}$ are linearly dependent, we just need to show that we can solve $\mathbf{0} = x_1 \mathbf{w}_1 + x_2 \mathbf{w}_2$ without setting both $x_1 = 0$ and $x_2 = 0$. But

$$\mathbf{0} = x_1 \mathbf{w}_1 + x_2 \mathbf{w}_2 = x_1 \lambda \mathbf{v}_1 + x_2 \rho \mathbf{v}_1 = (x_1 \lambda + x_2 \rho)\, \mathbf{v}_1$$

but this we can achieve by setting $x_2 = -\left(\frac{\lambda}{\rho}\right) x_1$.

COROLLARY 9.4. *Any two bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $B' = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ of a subspace S of $\mathbb{R}^n$ have the same number of vectors.*

Proof. From the fact that both B and B' are bases for S we know

(i) $S = span\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right)$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly independent
(ii) $S = span\left(\mathbf{w}_1, \ldots, \mathbf{w}_\ell\right)$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ are linearly independent.

Consider S as $span\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right)$. If $\ell > k$, then by the preceding lemma, the vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ would have to be linearly dependent. But that contradicts (ii). So $\ell \leq k$.

Reversing the roles of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ in the preceding paragraph, we can similarly conclude $k \leq \ell$. But then

$$\ell \leq k \quad \text{and} \quad k \leq \ell \quad \Rightarrow \quad k = \ell$$

DEFINITION 9.5. *Let W be a subspace of $\mathbb{R}^n$. The number of elements in any basis for W is the* ***dimension*** *of W.*

THEOREM 9.6. *Existence and Determination of Bases*

(1) *Every subspace of W of $\mathbb{R}^n$ has a basis and* $\dim(W) \leq n$.
(2) *Every linearly independent set of vectors in $\mathbb{R}^n$ can be enlarged, if necessary, to become a basis for $\mathbb{R}^n$.*
(3) *If W is a subspace of $\mathbb{R}^n$ and* $\dim(W) = k$, *then*
    (a) *every linearly independent set of k vectors in W is a basis for W.*
    (b) *every set of k vectors in W that spans W is a basis for W.*

3. Bases for Subspaces Associated to Matrices

Let $\mathbf{A}$ be an $m \times n$ matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{mn} & \cdots & a_{mn} \end{bmatrix}$$

Recall that the **column space** of $\mathbf{A}$ is the subspace of $\mathbb{R}^m$ spanned by the columns of $\mathbf{A}$:

$$ColSp(\mathbf{A}) = span(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n) \subset \mathbb{R}^m$$

where i^{th} column vector $\mathbf{c}_i$ is defined by

$$(\mathbf{c}_i)_j \equiv a_{ji} \quad , \quad j = 1, \ldots, m$$

Recalll also that the **row space** of $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$:

$$RowSp(\mathbf{A}) = span(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m) \subset \mathbb{R}^n$$

where the i^{th} row vector is defined by

$$(\mathbf{r}_i)_j = a_{ij} \quad , \quad j = 1, \ldots, n$$

A priori there is no particular relationship between the column space of $\mathbf{A}$ and the row space of $\mathbf{A}$; indeed, they are not even subspaces of the same space.

LEMMA 9.7. *If a matrix $\mathbf{A}'$ is row equivalent to a matrix $\mathbf{A}$ then the row space of $\mathbf{A}'$ is equal to the row space of $\mathbf{A}$.*

Proof. First we note that row operations can be built up from row operartions of the following form

$$(1) \qquad R_{ij}(\lambda_1, \lambda_2) : \left\{ \begin{array}{ll} \mathbf{r}_i \to \mathbf{r}'_i = \lambda_1 \mathbf{r}_i + \lambda_2 \mathbf{r}_j & i = j \\ \mathbf{r}_i \to \mathbf{r}'_i = \mathbf{r}_i \quad , & i \neq j \end{array} \right. \quad , \quad \lambda_1 \neq 0$$

For example, the interchange of i^{th} and j^{th} rows can be carried out as

$$\left\{ \begin{array}{c} \mathbf{r}_i \\ \mathbf{r}_j \end{array} \right\} \xrightarrow{R_{ij}(1,1)} \left\{ \begin{array}{c} \mathbf{r}'_i = \mathbf{r}_i + \mathbf{r}_j \\ \mathbf{r}'_j = \mathbf{r}_j \end{array} \right\} \xrightarrow{R_{ji}(-1,1)} \left\{ \begin{array}{c} \mathbf{r}''_i = \mathbf{r}'_i = \mathbf{r}_i + \mathbf{r}_j \\ \mathbf{r}''_j = -\mathbf{r}'_j + \mathbf{r}'_i = \mathbf{r}_i \end{array} \right\} \xrightarrow{R_{ij}(1,-1)} \left\{ \begin{array}{c} \mathbf{r}'''_i = \mathbf{r}''_i + \mathbf{r}''_j = \mathbf{r}_j \\ \mathbf{r}'''_j = \mathbf{r}''_j = \mathbf{r}_i \end{array} \right\}$$

while the other two elementary row operations can be viewed as simply special cases of the row operation (1).

Now suppose $\mathbf{v}$ is a vector lying in the span of row vectors of $\mathbf{A}$. I will show that it also lies in the span of the row vectors of the matrix $\mathbf{A}'$ obtained by applying the row operation (1) to $\mathbf{A}$.

$$\begin{aligned} \mathbf{v} &\in span(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m) \quad \Rightarrow \quad \mathbf{v} = c_1 \mathbf{r}_1 + \cdots + c_i \mathbf{r}_i + \cdots + c_j \mathbf{r}_j + \cdots + c_m \mathbf{r}_m \\ &= c_1 \mathbf{r}'_1 + \cdots + c_i \left(\frac{1}{\lambda_1} (\mathbf{r}'_i - \lambda_2 \mathbf{r}_j) \right) + \cdots + c_j \mathbf{r}'_j + \cdots + c_m \mathbf{r}'_m \\ &= c_1 \mathbf{r}'_1 + \cdots + \left(\frac{c_i}{\lambda_1} \right) \mathbf{r}'_i + \cdots + \left(c_j - \frac{c_i \lambda_2}{\lambda_1} \right) \mathbf{r}'_j + \cdots + c_m \mathbf{r}'_m \\ &\in span(\mathbf{r}'_1, \ldots, \mathbf{r}'_m) \end{aligned}$$

Thus, the row spaces of $\mathbf{A}$ and $\mathbf{A}'$ are the same. If $\mathbf{A}'$ is row equivalent to $\mathbf{A}$, then by definition there must be a sequence of row operations that converts $\mathbf{A}$ into $\mathbf{A}'$.

$$\mathbf{A} \to \mathbf{A}^{(1)} \to \mathbf{A}^{(2)} \to \cdots \to \mathbf{A}^{(k)} = \mathbf{A}'$$

From the preceding paragraph, we know at each intermediate stage we have $RowSp\left(\mathbf{A}^{(i)}\right) = RowSp\left(\mathbf{A}^{(i+1)}\right)$ so we conclude

$$RowSp(\mathbf{A}') = RowSp(\mathbf{A})$$

LEMMA 9.8. *Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{A}'$ be its reduction to row echelon form. Then the non-zero rows of $\mathbf{A}'$ form a basis for the row space of $\mathbf{A}$.*

The basis idea underlying the proof of this lemma is best illustrated by an example. Suppose $\mathbf{A}$ is a 4×5 matrix that is row equivalent to the following matrix in reduced row-echelon form

$$\mathbf{A}'' = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly the span of the row vectors of $\mathbf{A}'$ is just the span of the first three row vectors (that is to say, the contribution of the last row to the row space of $\mathbf{A}$ is just $\mathbf{0}$). On the other hand, it's clear the only way we can satisfy

$$\mathbf{0} = c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3$$

is by taking $c_1 = c_2 = c_3 = 0$; because that's the only way to kill off the components of the total sum that come from the pivots of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ (that is, we can't force a cancellation of terms coming from two different rows because only the pivot row will have a non-zero entry in the component corresponding to a column with a pivot). Thus,

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 \quad \Rightarrow \quad c_1 = c_2 = c_3 = 0 \\ &\Rightarrow \quad \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\} \text{ is a basis for } span\,(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = RowSp\,(\mathbf{A}'') = RowSp\,(\mathbf{A}) \end{aligned}$$

However, this isn't quite the statement of the lemma. For the lemma says the row vectors of a matrix in (un-reduced) echelon form should be a basis for the row space of $\mathbf{A}$. However, we can conclude this simply by noting that

$$\begin{aligned} \dim\left(RowSp(\mathbf{A})\right) &= \text{number of vectors in a basis for } RowSp\,(\mathbf{A}) \\ &= \text{number of non-zero rows in reduced echelon-form } \mathbf{A}'' \text{ of } \mathbf{A} \\ &= \text{number of non-zero rows in an echelon-form } \mathbf{A}' \text{ of } \mathbf{A} \end{aligned}$$

But because the row vectors of the matrix in echelon-form span $RowSp\,(\mathbf{A})$, and because the number of these row vectors is the same as the dimension of $RowSp\,(\mathbf{A})$, we can use Statement 3(b) of Theorem 9.6 (at the end of Lecture 9) to conclude that the row vectors of $\mathbf{A}'$ form a basis for $RowSp\,(\mathbf{A})$.

LEMMA 9.9. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix and let* $\mathbf{A}'$ *be its reduction to row echelon form. Then the columns of* $\mathbf{A}$ *corresponding to the columns of* $\mathbf{A}'$ *containing the pivots of* $\mathbf{A}'$ *form a basis for the column space of* $\mathbf{A}$.

Sketch of Proof. To find a basis for the column space, it would suffice to find a subset of linearly independent column vectors that spanned the entire column space. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be the columns of $\mathbf{A}$. A dependence relation among the $\mathbf{w}_i$ would be a non-trivial solution of

$$c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n = \mathbf{0} \tag{1}$$

whose augmented matrix would be

$$\left[\begin{array}{cccc|c} a_{11} & \cdots & \cdots & a_{1n} & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} & 0 \end{array}\right]$$

But if $\mathbf{A}'$ is a matrix in reduced row-echelon form obtained from $\mathbf{A}$ by row reduction, then $[\mathbf{A}' \mid 0]$ will correspond to an equivalent set of equations (that is to say, it will have exactly the same solutions as (1)). But when a matrix is in reduced row-echelon form, the dependence relations among its columns is manifest: the columns with pivots are always distinct standard basis vectors and the columns without a pivot can always be expressed in terms of the columns with pivots.[1] Thus, dependent columns of $\mathbf{A}'$ will be the

[1]Consider the following matrix in reduced row echelon form.

$$\mathbf{A}' = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Notice that the columns with pivots (columns 1, 2 and 4) are just the standard basis vectors $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ written vertically; and that the columns without pivots can be expressed as linear combinations of these standard basis vectors. For example, the fifth column can be written as

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

columns that don't contain pivots, and the columns that do contain pivots will be the linearly independent columns of $\mathbf{A}'$. But since the dependence relations among the columns of $\mathbf{A}'$ must be the same as the dependence relations among the columns of $\mathbf{A}$ (because the augmented matrices of each correspond to equivalent sets of equations), the independent columns of $\mathbf{A}$ will correspond to the columns of $\mathbf{A}'$ that contain pivots.

Let's demonstrate how this lemma is applied by an example.

EXAMPLE 9.10. Find a basis for the column space of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

- First we row reduce $\mathbf{A}$ to row-echelon form

$$\xrightarrow{\substack{R_1 \leftrightarrow R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 + R_3}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_3}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is a row-echelon form of $\mathbf{A}$. It has pivots in the 1^{st}, 2^{nd}, and 3^{rd} columns. Therefore, the 1^{st}, 2^{nd}, and 3^{rd} columns of the original matrix $\mathbf{A}$ will form a basis for the column space of $\mathbf{A}$:

$$ColSp\,(\mathbf{A}) = span\left(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}\right)$$

4. The Rank of a Matrix

THEOREM 9.11. *Let $\mathbf{A}$ be an $m \times n$ matrix. The dimension of the row space of $\mathbf{A}$ is equal to the dimension of its column space.*

This follows easily from the preceding two lemmas since the number of non-zero rows in a matrix in row-echelon form is exactly equal to the number of columns containing pivots. This theorem leads to the following definition.

DEFINITION 9.12. *The **rank** of a matrix is the dimension of its row space (which equals the dimension of its column space).*

Recall that the null space an $m \times n$ matrix $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ corresponding to the solution space of $\mathbf{Ax} = \mathbf{0}$.

THEOREM 9.13. *Let $\mathbf{A}$ be an $m \times n$ matrix. Then*

$$n = [\textit{number of columns of } \mathbf{A}] = \dim\left[\textit{Null space of } \mathbf{A}\right] + rank\,(\mathbf{A})$$

To see why this theorem must be true, consider the following simple example.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the fifth column depends on columns 1, 2, and 4. It turns out that if $\mathbf{A}'$ is in reduced row echelon form, then its columns without pivots can always be expressed as linear combinations of its columns with pivots.

This matrix is already in reduced row-echelon form. It has three pivots so

$$rank(\mathbf{A}) = \dim\left(RowSp(\mathbf{A})\right) = \dim\left(ColSp(\mathbf{A})\right) = 3$$

The dimension of its null space is evidently 1 since the solution of the corresponding homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ implies

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}\right.$$

but leaves x_4 undetermined. Hence, the dimension of the null space of $\mathbf{A}$ is 1. Thus,

$$4 = \text{ number of columns of } \mathbf{A} = 3 + 1 = (\text{rank of } \mathbf{A}) + (\dim\left(\text{null space of } \mathbf{A}\right))$$

In the next lecture we shall develop a geometric interpretation of this fundamental fact.