LECTURE 10

Linear Transformations

1. Functions between Sets

Let $\mathbf{A}$ be an $m \times n$ matrix. The goal of this lecture is to develop a geometric interpretation for homogeneous linear systems of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$.

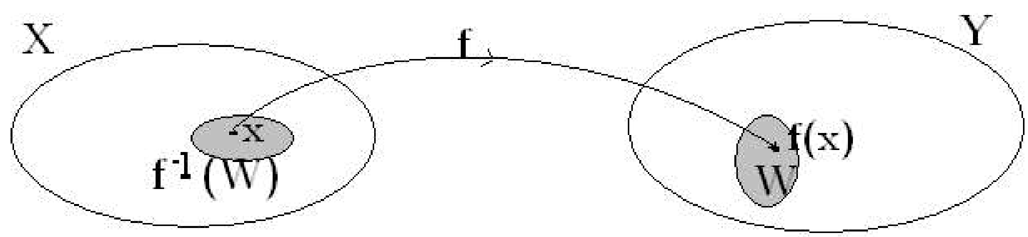
First let me recall some basic notions about maps between two sets. Let X and Y be sets. A **function** $f : X \to Y$ is a rule that associates with each element $x \in X$ an element $f(y) \in Y$. The set X is called the **domain** of the function f and the set Y is called the **codomain** (or target set) of f. The set

$$\{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

is called the **image** of the function f, and if W is a subset of Y, then the set

$$f^{-1}(W) = \{x \in X \mid f(x) \in W\}$$

is called the **inverse image of** W under f.



2. Linear Transformations

We shall now restrict our attention to the following kinds of maps.

Definition 10.1. *A function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if it satisfies*

(1) $\mathbf{T}(\mathbf{u} + \mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(\mathbf{v})$ *(i.e. the function $\mathbf{T}$ preserves vector addition)*
(2) $\mathbf{T}(r\mathbf{v}) = r\mathbf{T}(\mathbf{v})$ *(i.e., the function $\mathbf{T}$ preserves scalar multiplication)*

for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars $r \in \mathbb{R}$.

It is easy to see that if a mapping preserves both vector addition and scalar addition, then it will also preserve any combination of such operations; that is to say, it will preserve arbitrary linear combinations of vectors

$$\mathbf{T}(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1\mathbf{T}(\mathbf{v}_1) + r_2\mathbf{T}(\mathbf{v}_2) + \cdots + r_k\mathbf{T}(\mathbf{v}_k)$$

EXAMPLE 10.2. Show that the transformation $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^3 : (s,t) \to (t,s,1+t+s)$ is not a linear transformation.

- Let $\mathbf{v} = (s,t)$ Then

$$\begin{aligned} \mathbf{T}(\mathbf{v}) &= \mathbf{T}(s,t) = (t,s,1+t+s) \\ \mathbf{T}(r\mathbf{v}) &= \mathbf{T}(rs,rt) = (rt,rs,1+rs+rt) \\ &\neq r(t,s,1+t+s) = r\mathbf{T}(\mathbf{v}) \end{aligned}$$

and so $\mathbf{T}$ does not preserve scalar multiplication: hence it is not a linear transformation.

EXAMPLE 10.3. Let $\mathbf{A}$ be an $n \times m$ matrix. To any vector in $\mathbb{R}^m$, we can associate an $m \times 1$ column vector $\mathbf{x}$, and via multiplication from the left by $\mathbf{A}$, a $n \times 1$ column vector

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m \end{bmatrix} \in \mathbb{R}^n$$

Define $T_A : \mathbb{R}^m \to \mathbb{R}^n$ by

$$T_A(\mathbf{x}) = \mathbf{Ax}$$

Then

$$\begin{aligned} T(\lambda\mathbf{x}) &= \mathbf{A}(\lambda\mathbf{x}) = \lambda\mathbf{Ax} = \lambda T(\mathbf{x}) \\ T(\mathbf{x}_1 + \mathbf{x}_2) &= \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}x_1 + \mathbf{A}\mathbf{x}_2 = T(\mathbf{x}_1) + T(\mathbf{x}_2) \end{aligned}$$

and so $T_{\mathbf{A}}$ is a linear transformation.

3. Linear Transformations and Matrices

The last example says that to an $n \times m$ matrix $\mathbf{A}$we can also associate a linear transformation $T_{\mathbf{A}} : \mathbb{R}^m \to \mathbb{R}^n$. We shall now show that the converse is also true: to every linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ we can associate an $n \times m$ matrix $\mathbf{A}_T$ such that

$$T(\mathbf{x}) = \mathbf{A}_T\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^m \quad .$$

LEMMA 10.4. *Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and let $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$. Then every vector in the image of $\mathbf{T}$ can be written as a linear combination of the vectors $\mathbf{T}(\mathbf{b}_1), \mathbf{T}(\mathbf{b}_2), \ldots, \mathbf{T}(\mathbf{b}_n)$.*

Proof. Since B is a basis for $\mathbb{R}^n$, any vector $\mathbf{v} \in \mathbb{R}^n$ can be expressed as

$$\mathbf{v} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_n\mathbf{b}_n$$

And so the image of a vector $\mathbf{v}$ by $\mathbf{T}$ will be expressible as

$$\begin{aligned} \mathbf{T}(\mathbf{v}) &= \mathbf{T}(r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_n\mathbf{b}_n) \\ &= r_1\mathbf{T}(\mathbf{b}_1) + r_2\mathbf{T}(\mathbf{b}_2) + \cdots + r_k\mathbf{T}(\mathbf{b}_k) \quad \text{(since } \mathbf{T} \text{ is a linear transformation)} \end{aligned}$$

□

THEOREM 10.5. *Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, let $\{\mathbf{e}_i \mid i = 1, \ldots, n\}$ be the standard basis for $\mathbb{R}^n$:*

$$(\mathbf{e}_i)_j = \begin{cases} 1 & , \quad j = i \\ 0 & , \quad j \neq i \end{cases}$$

and let $\mathbf{A}$ be the $m \times n$ matrix whose i^{th} column coincides with $\mathbf{T}(\mathbf{e}_i) \in \mathbb{R}^m$. Then

$$\mathbf{T}(\mathbf{x}) = \mathbf{Ax}$$

In other words, every linear transformation $\mathbf{T}:\mathbb{R}^n\to\mathbb{R}^m$ is equivalent to the matrix mulitiplication of the vectors $\mathbf{x}\in\mathbb{R}^n$ by an $m\times n$ matrix $\mathbf{A}$. The converse of this fact is also true, if $\mathbf{A}$ is an $m\times n$ matrix and $\mathbf{T}:\mathbb{R}^n\to\mathbb{R}^m$ is the mapping defined by

$$\mathbf{x}\in\mathbb{R}^n\to\mathbf{Ax}\in\mathbb{R}^m$$

then $\mathbf{T}$ is a linear transformation.

EXAMPLE 10.6. Find the matrix corresponding to the linear transformation $\mathbf{T}:\mathbb{R}^2\to\mathbb{R}^3$ given by $\mathbf{T}(x_1,x_2)=(x_1-x_2,x_1+x_2,x_1)$.

- We have

$$\begin{aligned}\mathbf{T}(\mathbf{e}_1) &= \mathbf{T}(1,0)=(1-0,1+0,1)=(1,1,1)\\ \mathbf{T}(\mathbf{e}_2) &= \mathbf{T}(0,1)=(0-1,0+1,0)=(-1,1,0)\end{aligned}$$

Hence

$$\mathbf{A}=[\mathbf{T}(\mathbf{e}_1),\mathbf{T}(\mathbf{e}_2)]=\begin{bmatrix}1&-1\\1&1\\1&0\end{bmatrix}$$

We confirm

$$\mathbf{Ax}=\begin{bmatrix}1&-1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1-x_2\\x_1+x_2\\x_1\end{bmatrix}=\mathbf{Tx}$$

4. Subspaces associated with linear transformations

Recall that we had three natural subspaces associated with a matrix $\mathbf{A}$; its row space, its column space and its null space. As we have just seen, every linear transformation corresponds to a matrix, and so there should also be three natural subspaces associated to a linear transformation T. This turns out to be the case, but we shall postpone the connection with matrices for the time being, and instead give some more intrinsic definitions.

DEFINITION 10.7. *The* ***kernel*** *of a linear transformation* $\mathbf{T}:\mathbb{R}^n\to\mathbb{R}^m$ *is the set of all* $\mathbf{x}\in\mathbb{R}^n$ *such that* $\mathbf{Tx}=\mathbf{0}\in\mathbb{R}^m$.

$$\ker(T)=\{\mathbf{x}\in\mathbb{R}^m\mid T(\mathbf{x})=\mathbf{0}\}$$

LEMMA 10.8. *The kernel of a linear transformation* $\mathbf{T}:\mathbb{R}^n\to\mathbb{R}^m$ *is a subspace of* $\mathbb{R}^m$.

Proof. $\ker(T)$ is obviously a subset of $\mathbb{R}^m$. We need to show that it's closed under scalar multiplication and vector addition. Let $\lambda\in\mathbb{R}$ and $\mathbf{x}\in\ker(T)$ be arbitary elements of their respective sets. Then $T(\lambda\mathbf{x})=\lambda T(\mathbf{x})$, since T is a linear transformation. But $T(\mathbf{x})=0$ since $\mathbf{x}\in\ker(T)$. So $T(\lambda\mathbf{x})=\mathbf{0}$. We conclude that if $\lambda\in\mathbb{R}$, and $\mathbf{x}\in\ker(T)$, then $\lambda\mathbf{x}\in\ker(T)$, and so $\ker(T)$ is closed under scalar multiplication.

Now let $\mathbf{x}_1,\mathbf{x}_2$ be arbitrary vectors in $\ker(T)$. Then since T is a linear transformation, $T(\mathbf{x}_1+\mathbf{x}_2)=T(\mathbf{x}_1)+T(\mathbf{x}_2)=\mathbf{0}+\mathbf{0}=\mathbf{0}$ and so $\mathbf{x}_1+\mathbf{x}_2\in\ker(T)$. Thus, $\ker(T)$ is closed under vector addition.

Since $\ker(T)$ is a subset of $\mathbb{R}^m$ that is closed under both scalar multiplication and vector addition, it is a subspace of $\mathbb{R}^m$. □

DEFINITION 10.9. *The* ***image*** *or* ***range*** *of* $\mathbf{T}$ *is the set of all* $\mathbf{y}\in\mathbb{R}^m$ *such that* $\mathbf{y}=\mathbf{T}(\mathbf{x})$ *for some* $\mathbf{x}\in\mathbb{R}^n$.

$$\mathit{range}(T)=\{\mathbf{y}\in\mathbb{R}^n\mid\mathbf{y}=T(\mathbf{x})\quad\text{for some }\mathbf{x}\in\mathbb{R}^m\}$$

LEMMA 10.10. *The range of a linear transformation* $T:\mathbb{R}^m\to\mathbb{R}^n$ *is a subspace of* $\mathbb{R}^n$.

Proof. We need to show that $range\,(T)$ is closed under both scalar multiplication and vector addition.

Suppose $\mathbf{y} \in range\,(T)$. Then there must be an $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{y} = T\,(\mathbf{x})$. But then $\lambda\mathbf{x} \in \mathbb{R}^m$ and

$$T\,(\lambda\mathbf{x}) = \lambda T\,(\mathbf{x}) = \lambda\mathbf{y}$$

and so $\lambda\mathbf{y}$ is in $range\,(T)$. Hence, $range\,(T)$ is closed under scalar multiplication.

Suppose $\mathbf{y}_1, \mathbf{y}_2 \in range\,(T)$. Then there must be vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$ such that $\mathbf{y}_1 = T\,(\mathbf{x}_1)$ and $\mathbf{y}_2 = T\,(\mathbf{x}_2)$. Now apply T to the vector sum $\mathbf{x}_1 + \mathbf{x}_2$:

$$T\,(\mathbf{x}_1 + \mathbf{x}_2) = T\,(\mathbf{x}_1) + T\,(\mathbf{x}_2) = \mathbf{y}_1 + \mathbf{y}_2$$

This displays $\mathbf{y}_1 + \mathbf{y}_2$ as an element of $range\,(T)$.

Since $range\,(T) \subset \mathbb{R}^n$ is closed under both scalar multiplication and vector addition, it is a subspace of $\mathbb{R}^n$. $\square$

Now let $\mathbf{A}$ be the $m \times n$ matrix corresponding to a linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$\begin{aligned} \ker(\mathbf{T}) &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{T}\,(\mathbf{x}) = \mathbf{0}\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}_T\mathbf{x} = \mathbf{0}\} = \text{ Null space of } \mathbf{A}_T \end{aligned}$$

$$\begin{aligned} range\,(\mathbf{T}) &= \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = \mathbf{T}\,(\mathbf{x}) \quad , \quad \text{for some } \mathbf{x} \in \mathbb{R}^n\} \\ &= \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = \mathbf{A}_T\mathbf{x} \quad , \quad \text{for some } \mathbf{x} \in \mathbb{R}^n\} = \text{ column space of } \mathbf{A}_T \end{aligned}$$

Note also that the dimension n of the domain $\mathbb{R}^n$ of $\mathbf{T}$ is same as the number of columns in the corresponding matrix $\mathbf{A}$. Now from Theorem 10.6 of Lecture 10 we know

$$(\text{number of columns of } \mathbf{A}) = (\text{dimension of null space of } \mathbf{A}) + (\text{dimension of column space of } \mathbf{A})$$

In terms of notions of linear transformations this translates to

$$(\text{dimension of domain of } \mathbf{T}) = (\text{dimension of kernel of } \mathbf{T}) + (\text{dimension of range of } \mathbf{T})$$

EXAMPLE 10.11. Consider the linear transformation $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$\mathbf{T}\,(x_1, x_2, x_3) = (x_1 + x_3, x_1 + x_2 + 2x_3, -x_1 + x_2, 2x_2 + 2x_3)$$

Find a basis for the kernel of $\mathbf{T}$ and a basis for the range of $\mathbf{T}$.

- Let's first find the matrix representation of $\mathbf{T}$. We have

  $$\begin{aligned} \mathbf{T}\,(\mathbf{e}_1) &= \mathbf{T}\,(1,0,0) = (1+0, 0+0+2(0), -1+1, 2(0)+2(0)) = (1,1,-1,0) \\ \mathbf{T}\,(\mathbf{e}_2) &= \mathbf{T}\,(0,1,0) = (0+0, 0+1+2(0), -0+1, 2(1)+2(0)) = (0,1,1,2) \\ \mathbf{T}\,(\mathbf{e}_3) &= \mathbf{T}\,(0,0,1) = (0+1, 0+0+2(1), -0+0, 2(0)+2(1)) = (1,2,0,2) \end{aligned}$$

  and so the linear transformation $\mathbf{T}$ corresponds to the 4×3 matrix

  $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

  As we pointed out above the kenel of $\mathbf{T}$ is the same as the null space of $\mathbf{A}$ and the range of $\mathbf{T}$ is the same thing as the column space of $\mathbf{A}$. To find the null space and column space of a matrix we first row reduce $\mathbf{A}$ to reduced row-echelon form

  $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{A}'$$

From Lecture 10 (Lemma 10.3) we know that the column space of the matrix $\mathbf{A}$ is spanned by the columns in $\mathbf{A}$ which correspond to columns in the row-echelon form $\mathbf{A}'$ that contain pivots. Thus,

$$\text{range of } \mathbf{T} = \text{column space of } \mathbf{A} = \text{ span}\left(\left[\begin{array}{c} 1 \\ 1 \\ -1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 2 \end{array}\right]\right)$$

The kernel of $\mathbf{T}$ can be identified with the null space of $\mathbf{A}$, which is equal to the null space of $\mathbf{A}'$: i.e, the solution set

$$\left.\begin{array}{c} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = -x_3 \\ x_2 = -x_3 \end{array}\right.$$

so

$$\mathbf{x} = \left[\begin{array}{c} -x_3 \\ -x_3 \\ x_3 \end{array}\right] = x_3 \left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]$$

So

$$\ker(\mathbf{T}) = \textit{span}\left(\left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]\right)$$

5. Composition of Linear Transformations

Suppose we have two linear transformations

$$\begin{array}{rcl} \mathbf{T}_1 & : & \mathbb{R}^n \to \mathbb{R}^m \\ \mathbf{T}_2 & : & \mathbb{R}^m \to \mathbb{R}^p \end{array}$$

Because every element in the range of $\mathbf{T}_1$ can be regarded as an element in the domain of $\mathbf{T}_2$ the composed mapping

$$\mathbf{T}_2 \circ \mathbf{T}_1 : \mathbb{R}^n \to \mathbb{R}^p \quad ; \quad \mathbf{x} \in \mathbb{R}^n \quad \mapsto \quad \mathbf{T}_2(\mathbf{T}_1(\mathbf{x})) \in \mathbb{R}^p$$

is well defined, and, in fact, is another linear transformation. Indeed, if we switch back to our matrix language, where the transformations $\mathbf{T}_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{T}_2 : \mathbb{R}^m \to \mathbb{R}^p$ are implemented by, respectively, an $m \times n$ matrix $\mathbf{A}_1$ and an $p \times m$ matrix $\mathbf{A}_2$, then to the composed transformation $\mathbf{T}_2 \circ \mathbf{T}_1 : \mathbb{R}^n \to \mathbb{R}^p$ we have the following matrix:

$$\mathbf{A}_{12} = \mathbf{A}_2 \mathbf{A}_1$$

Note that this matrix multiplication is also well-defined since the number m of columns of $\mathbf{A}_2$ is the same as the number m of rows of $\mathbf{A}_1$.

Example 10.12. Consider the linear transformation corresponding to a rotation in the xy plane by an angle θ

$$\begin{array}{rcl} x & \to & x' = x\cos(\theta) + y\sin(\theta) \\ y & \to & y' = -x\sin(\theta) + y\cos(\theta) \end{array}$$

To this linear transformation corresponds the following 2×2 matrix:

$$\mathbf{A} = \left[\begin{array}{cc} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array}\right]$$

If we apply this transformation twice, the effect should be that of a two rotations by the angle θ. We thus should have

$$\mathbf{AA} = \left[\begin{array}{cc} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{array}\right] \tag{10.1}$$

Calculating the matrix multiplication on the left hand side:

$$\mathbf{AA} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\cos(\theta)\sin(\theta)) \\ -2\cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix} \tag{10.2}$$

Comparing (10.1) with (10.2) we see we must have

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) &= \cos(\theta)\sin(\theta) \end{aligned}$$

We have thus, by a simple matrix calculation, rederived the double angle trig identities one learns in high school.