LECTURE 12

Determinants

1. Determining when a square matrix has an inverse

Thus far, the quickest way we have to determine if a given (square) matrix has an inverse or not is by row-reducing the matrix to a row-echelon form and seeing whether or not every column of the row-echelon form has a pivot. This method is not very satisfactory because if you want to consider a set of matrices you have to apply the above algorithm to each matrix individually. What we develop in this lecture is a simple function of the entries of a matrix whose value will tell you whether or not the matrix is invertible. In what follows we will denote by M_n the set of $n \times n$ matrices.

Let's start with the simplest case; that of a 1×1 matrix. Clearly,

$$\mathbf{A} = [a]$$

is invertible if and only if $a \neq 0$. If define a function $\det : M_1 \to \mathbb{R} \quad , \quad \det([a]) = a$. then $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) = 0$. (Do not be put off by the apparent tautology of this example, its generalization will be much more substantial.)

Now let's consider the case of a generic 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's try to row reduce this matrix to row echelon form.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \xrightarrow{R_2 \to R_2 - \frac{c}{a} R_1} \quad \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix}$$

Note that if $d - \frac{c}{a} b = 0$, or equivalently, if $ad - bc \neq 0$, the matrix will not be row reducible to the identity matrix (because the last row in its row echelon form will be a zero roow). On the other hand, if $ad - bc \neq 0$, then the matrix can be row-reduced to the identity matrix and hence invertible. So if we set

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

then a 2×2 matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.

In this lecture we shall define a function $\det : M_n \to \mathbb{R}$ of the entries of an $n \times n$ matrix which will have the property that

$$\det(\mathbf{A}) \neq 0 \qquad \Leftrightarrow \qquad \mathbf{A} \text{ is invertible}$$

2. General Determinants

The formula for the determinant of a general $n \times n$ will in general involve $n!$ separate terms; thus, the determinant for a 4×4 matrix will involve $4! = 24$ terms, and the determinant for a 5×5 matrix will involve $5! = 60$ terms! Rather than giving an explicit formula for the higher order determinants, we'll present an recursive definition.

DEFINITION 12.1. *Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The **minor matrix** corresponding to the entry a_{ij} is the $(n-1) \times (n-1)$ matrix formed from $\mathbf{A}$ by deleting the i^{th} row and j^{th} column of $\mathbf{A}$.*

DEFINITION 12.2. *The determinant of a 1×1 matrix is its sole entry. Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The **cofactor** C_{ij} of a_{ij} is $(-1)^{i+j}$ times the determinant of the minor matrix M_{ij} corresponding to a_{ij}.*

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

The determinant of $\mathbf{A}$ is

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

EXAMPLE 12.3. Calculate the determinant of the following matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- We have

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (1)\begin{vmatrix} 0 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} - (2)\begin{vmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} + (0)\begin{vmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} - (1)\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

Now

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (0)\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - (1)\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + (2)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 0 - (0-2) + 2(1+1) = 6$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - (1)\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} + (2)\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = -(-2) - (-2) + 2(1) = 6$$

$$\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (0)\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(1+1) + 0 + (-1) = -3$$

So

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (1)(6) - (2)(6) + 0 - (-3) = -3$$

3. An even more general formula for $\det(\mathbf{A})$

In the recursive definition of the determinant we have an expansion of the determinant in terms of the entries and minors of the first row of the matrix. There is nothing special about the first row, however. One can also express the determinant of the matrix in terms of the entries and minors of any of its rows. In fact, one can express the determinant of a matrix in terms of the entries and minors of any of its columns.

DEFINITION 12.4. *Let $\mathbf{A}$ be an $n \times n$ matrix. The ij^{th} **cofactor** C_{ij} is $(-1)^{i+j}$ times the determiant of the ij^{th} minor of $\mathbf{A}$*

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

THEOREM 12.5. *Suppose $\mathbf{A}$ is an $n \times n$ matrix with entries $\{a_{ij}\}_{\substack{1 \le i \le n \\ 1 \le j \le n}}$*

- *If i is any row index of* $\mathbf{A}$ *then*

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij}) \qquad \textit{(sums are over the column index } j\textit{)}$$

- *If j is any column index of* $\mathbf{A}$ *then*

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij}) \qquad \textit{(sums are over the row index } i\textit{)}$$

These formulas are convenient to use when a particular row or column of $\mathbf{A}$ has a lot of zeros.

EXAMPLE 12.6. Compute $\det(\mathbf{A})$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Notice that the second column has only one non-zero entry. We'll choose to expand the determinant along that column.

$$\begin{aligned} \det(\mathbf{A}) &= a_{12}(-1)^{1+2}\det(M_{12}) + a_{22}(-1)^{2+2}\det(M_{22}) + a_{32}(-1)^{3+2}\det(M_{32}) + a_{42}(-1)^{4+2}\det(M_{42}) \\ &= (1)(-1)^{3}\det(M_{12}) + (0)(-1)^{2+2}\det(M_{22}) + (0)(-1)^{3+2}\det(M_{32}) + (0)(-1)^{4+2}\det(M_{42}) \\ &= -\det(M_{12}) = -\det\left(\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}\right) \end{aligned}$$

To compute that last determinant it is convenient to expand along the bottom row

$$\begin{aligned} \det\left(\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}\right) &= (1)(-1)^{3+1}\det(M_{31}) + (0)(-1)^{3+2}\det(M_{32}) + (0)(-1)^{3+3}\det(M_{33}) \\ &= \det(M_{31}) = \det\left(\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\right) = 2-(-2) = 4 \end{aligned}$$

We conclude that

$$\det(\mathbf{A}) = -\det\left(\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\right) = -4$$

4. Computing determinants by row reduction

The following theorem tells us how the elementary row operations affect the determinant of a matrix.

THEOREM 12.7. *Suppose* $\mathbf{A}$ *is an* $n \times n$ *matrix*

(1) *If* $\mathbf{A}'$ *is a matrix obtained from* $\mathbf{A}$ *by interchanging two of its rows then* $\det(\mathbf{A}') = -\det(\mathbf{A})$.
(2) *If* $\mathbf{A}$ *is a square matrix and* $\mathbf{A}'$ *is a matrix obtained from* $\mathbf{A}$ *by multiplying one of its rows by a scalar* r *then* $\det(\mathbf{A}') = r\det(\mathbf{A})$
(3) *If* $\mathbf{A}$ *is a square matrix and* $\mathbf{A}'$ *is a matrix obtained from* $\mathbf{A}$ *by adding a scalar multiple of one row to another, then* $\det(\mathbf{A}') = \det(\mathbf{A})$.

The above theorem tells us what effect each of the elementary row operations has on the determinant of a matrix. We also have

LEMMA 12.8. *If* $\mathbf{A}$ *is an* $n \times n$ *matrix in row echelon form then* $\det(\mathbf{A})$ *is equal to the products of the entries along the diagonal.*

We won't prove this theorem in general. However, the basic idea underlying this theorem is easily demonstrated by a simple example.

EXAMPLE 12.9. Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

We have

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} &= (1)\begin{vmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{vmatrix} - (2)\begin{vmatrix} 0 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{vmatrix} + (3)\begin{vmatrix} 0 & 5 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 10 \end{vmatrix} - (4)\begin{vmatrix} 0 & 5 & 6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{vmatrix} \\
&= (1)\left((5)\begin{vmatrix} 8 & 9 \\ 0 & 10 \end{vmatrix} - (6)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} + (7)\begin{vmatrix} 0 & 8 \\ 0 & 0 \end{vmatrix}\right) \\
&\quad -(2)\left((0)\begin{vmatrix} 8 & 9 \\ 0 & 10 \end{vmatrix} - (6)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} + (7)\begin{vmatrix} 0 & 8 \\ 0 & 0 \end{vmatrix}\right) \\
&\quad +(3)\left((0)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} - (5)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} + (7)\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\right) \\
&\quad -(4)\left((0)\begin{vmatrix} 0 & 8 \\ 0 & 0 \end{vmatrix} - (5)\begin{vmatrix} 0 & 0 \\ 0 & 8 \end{vmatrix} + (6)\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\right) \\
&= (1)\left((5)(8)(10) + (6)(0) + (7)(0)\right) - (2)\left((0)(8)(9) - (6)(0) + (7)(0)\right) \\
&\quad +(3)\left((0)(0) - (5)(0) + (7)(0)\right) - (4)\left((0)(0) - (5)(0) + (6)(0)\right) \\
&= (1)(5)(8)(10) + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\
&= (1)(5)(8)(10)
\end{aligned}$$

Combining the theorem and lemma above we can now conclude

COROLLARY 12.10. *If* $\mathbf{A}$ *is an* $n \times n$ *matrix and* $\mathbf{A}'$ *is a row-echelon form of* $\mathbf{A}$ *obtained without row rescalings, then*

$$\det(\mathbf{A}) = (-1)^j \prod_{i=1}^{n} a'_{ii}$$

where j *is the total number of row-interchanges that occured in row-reducing* $\mathbf{A}$ *to* $\mathbf{A}'$. *Here* $\prod_{\iota=1}^{n} a'_{ii}$ *is the total product of the diagonal entries of* $\mathbf{A}'$.

COROLLARY 12.11. *A square matrix* $\mathbf{A}$ *is invertible if and only if* $\det(\mathbf{A}) \neq 0$.

THEOREM 12.12. *If* $\mathbf{A}$ *is a square matrix and* $\mathbf{A}^T$ *is its transpose, then*

$$\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$$

THEOREM 12.13. *The determinant of a product of two square matrices is equal to the product of their determinants:*

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}).$$

THEOREM 12.14 (Crammer's Rule). *Suppose* $\mathbf{A}$ *is an invertible* $n \times n$ *matrix and* $\mathbf{b} \in \mathbb{R}^n$. *Then the (unique) solution to the* $n \times n$ *linear system* $\mathbf{Ax} = \mathbf{b}$ *is given by*

$$x_i = \frac{\det(\mathbf{B}_i)}{\det(\mathbf{A})} \qquad i = 1, \ldots, n$$

where $\mathbf{B}_i$ *is the* $n \times n$ *matrix obtained from* $\mathbf{A}$ *by replacing the* i^{th}*-column of* $\mathbf{A}$ *with the column vector* $\mathbf{b}$.

EXAMPLE 12.15. Use Crammer's Rule to solve

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

We have

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} && \Rightarrow \quad \det(\mathbf{A}) = 3+2 = 5 \\ \mathbf{B}_1 &= \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} && \Rightarrow \quad \det(\mathbf{B}_1) = -3+8 = 5 \\ \mathbf{B}_2 &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} && \Rightarrow \quad \det(\mathbf{B}_2) = 3+2 = 5 \end{aligned}$$

So the solution of $\mathbf{Ax} = \mathbf{b}$ will be

$$x_1 = \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{5}{5} = 1 \qquad , \qquad x_2 = \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = \frac{-5}{5} = -1$$

And sure enought $\mathbf{x} = [1, -1]$ solves the original linear system.

THEOREM 12.16.

DEFINITION 12.17. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. The* ***cofactor matrix*** $\mathbf{A}'$ *of* $\mathbf{A}$ *is the* $n \times n$ *matrix whose entries consists of the cofactors of* $\mathbf{A}$*:*

$$(\mathbf{A}')_{ij} = C_{ij} = (-1)^{i+j}\det(M_{ij})$$

DEFINITION 12.18. *The* ***adjoint*** *of an* $n \times n$ *matrix* $\mathbf{A}$ *is the* $n \times n$ *matrix defined by*

$$adj(\mathbf{A}) = (\mathbf{A}')^T$$

THEOREM 12.19. *Let* $\mathbf{A}$ *be an invertible matrix, then*

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A})$$

EXAMPLE 12.20. Use the preceding theorem to calculate the inverse of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & 1 & -1 \end{bmatrix}$$

We first need to determine the cofactors of $\mathbf{A}$

$$\begin{aligned} C_{11} &= (-1)^{1+1}\det\left(\begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}\right) = -7 \\ C_{12} &= (-1)^{1+2}\det\left(\begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}\right) = +3 \\ C_{13} &= (-1)^{1+3}\det\left(\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\right) = +3 \end{aligned}$$

$$\begin{aligned} C_{21} &= (-1)^{2+1}\det\left(\begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}\right) = +5 \\ C_{22} &= (-1)^{2+2}\det\left(\begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}\right) = -2 \\ C_{23} &= (-1)^{2+3}\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right) = -2 \end{aligned}$$

$$\begin{aligned} C_{31} &= (-1)^{3+1} \det \left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \right) = -3 \\ C_{32} &= (-1)^{3+2} \det \left(\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \right) = +2 \\ C_{33} &= (-1)^{3+3} \det \left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \right) = 1 \end{aligned}$$

So the cofactor matrix is

$$\mathbf{C} = \begin{bmatrix} -7 & 3 & 3 \\ 5 & -2 & -2 \\ -3 & 2 & 1 \end{bmatrix}$$

Its transpose is

$$\mathbf{C}^T = \begin{bmatrix} -7 & 5 & -3 \\ 3 & -2 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

Also

$$\det(\mathbf{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (2)(-7) + (1)(3) + (4)(3) = -14 + 3 + 12 = 1$$

According to the theorem

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \frac{1}{1} \begin{bmatrix} -7 & 5 & -3 \\ 3 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

and sure enought

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -7 & 5 & -3 \\ 3 & -2 & 2 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Application: Calculating the Area of a Parallelogram

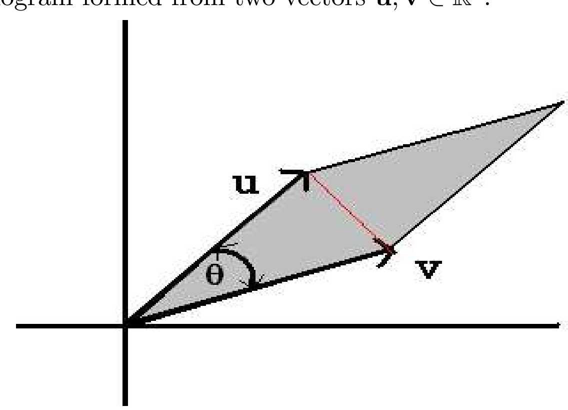
DEFINITION 12.21. *Let* $\mathbf{A}$ *be a* 2×2 *matrix.*

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

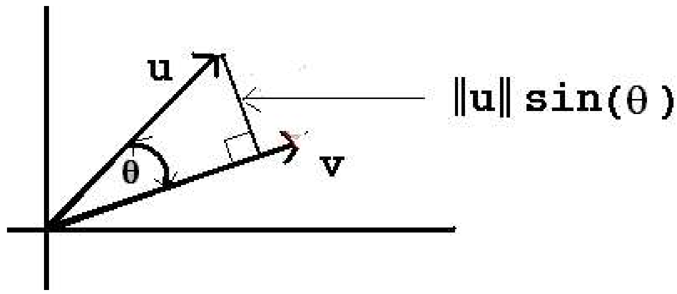
The determinant of $\mathbf{A}$ *is the number*

$$\det(\mathbf{A}) = ad - bc$$

Now consider the parallelogram formed from two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.



The area of this parallelogram is evidently the sum of the areas the two isometric triangles obtained by bisecting the parallelogram along the line from the tip of $\mathbf{u}$ to the tip of $\mathbf{v}$. Now the height of a triangle formed from $\mathbf{u}$ and $\mathbf{v}$ will be given by $\|\mathbf{u}\| \sin(\theta)$.



The area of this triangle is given by a formula from high school geometry:

$$\begin{aligned} \text{area of a triangle} &= \frac{1}{2}(\text{length of base}) \times (\text{height of triangle}) \\ &= \frac{1}{2}\|\mathbf{v}\|\|\mathbf{u}\|\sin(\theta) \end{aligned}$$

Hence, the area for the parallelogram will be

$$\begin{aligned} \text{area of parallelogram} &= 2 \text{ times area of } \Delta_{\mathbf{uv}} \\ &= 2\cdot\frac{1}{2}\|\mathbf{v}\|\|\mathbf{u}\|\sin(\theta) \\ &= \|\mathbf{v}\|\|\mathbf{u}\|\sin(\theta) \\ &= \|\mathbf{v}\|\|\mathbf{u}\|\sqrt{1-\cos^2(\theta)} \end{aligned}$$

If we square both sides we have

$$\begin{aligned} area^2 &= \|\mathbf{v}\|^2\|\mathbf{u}\|^2\left(1-\cos^2(\theta)\right) \\ &= \|\mathbf{v}\|^2\|\mathbf{u}\|^2\left(1-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{u}\|}\right)^2\right) \\ &= \|\mathbf{v}\|^2\|\mathbf{u}\|^2-(\mathbf{u}\cdot\mathbf{v})^2 \\ &= (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\mathbf{v})^2 \\ &= \left(v_1^2+v_2^2\right)\left(u_1^2+u_2^2\right)-(u_1v_1+u_2v_2)^2 \\ &= v_1^2u_1^2+v_1^2u_2^2+v_2^2u_1^2+v_2^2u_2^2-v_1^2u_1^2-2u_1v_1u_2v_2-v_2^2u_2^2 \\ &= v_1^2u_2^2+v_2^2u_1^2-2u_1v_1u_2v_2 \\ &= (u_1v_2-u_2v_1)^2 \end{aligned}$$

So

$$area = |u_1v_2 - u_2v_1|$$

Now let $\mathbf{A}$ be the 2×2 matrix formed by interpreting $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ as its columns:

$$\mathbf{A} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

Evidently,

$$|\det(\mathbf{A})| = |u_1v_2 - u_2v_1| = \text{area of parallelogram formed from } \mathbf{u} \text{ and } \mathbf{v}$$

6. Calculating the Volume of a Parallelopiped

DEFINITION 12.22. *Let* $\mathbf{A}$ *be a* 3×3 *matrix.*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of $\mathbf{A}$ *is the number*

$$\begin{aligned}\det(\mathbf{A}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\end{aligned}$$

A **parallelopiped** is the 3-dimensional analog a 2-dimensional parallelogram; these are constructed by regarding three 3-dimensional vectors and their translates as the edges of a solid body, and the parallelograms formed from pairs of these vectors as the sides. We shall not work out the geometry of this example but simply state the following fact:

FACT 12.23. *Let* $P_{\mathbf{a},\mathbf{b},\mathbf{c}}$ *be the parallelopiped associated with three vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$*, and let* $\mathbf{A}$ *be the* 3×3 *matrix formed by regarding the vectors* $\mathbf{a}, \mathbf{b}$*, and* $\mathbf{c}$ *as its column vectors. Then*

$$\text{volume of } P_{\mathbf{a},\mathbf{b},\mathbf{c}} = |\det(\mathbf{A})|$$