Math 3013

SOLUTIONS TO SAMPLE FIRST EXAM

1. Let

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Compute the matrix product **BC**

$$\mathbf{BC} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & -1 & 3 \end{bmatrix}$$

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2. For each of the following augmented matrices, indicate

- the number of equations and the number of variables in the corresponding linear system
- whether or not the corresponding linear system has a solution
- if the corresponding linear system does have a solution, the number of free variables in the solution.

Hint: note that these augmented matrices are already in row echelon form.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- This augmented matrix comes from a system of 4 equations and 4 unknowns. There is a solution. Since there is one column without a pivot, there is exacly one free variable (x_3) in the solution.

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$

- This augmented matrix comes from a system of 3 equations in 4 unknowns. There is no solution since the third row correspond to the equation $0 = -1$.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- This augmented matrix comes from a system of 4 equations in 3 unknowns. There is a solution. Since there are no columns without pivots, there are no free parameters in the solution. The solution is therefore unique.

3. Consider the following linear system

$$\begin{aligned} -x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 3 \end{aligned}$$

Write down the corresponding augmented matrix and row reduce it to row-echelon form.

- The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ -1 & 1 & 1 & 3 \end{array}\right]$$

The first thing we need is a pivot in the upper left hand corner:

$$\xrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

This last matrix is in row echelon form.

4. Row reduce the following augmented matrix to **reduced** row-echelon form.

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccccc|c} 2 & 2 & 4 & 6 & 2 & 2 \\ 0 & 0 & 3 & 6 & 6 & 3 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{ccccc|c} 2 & 2 & 4 & 6 & 2 & 2 \\ 0 & 0 & 3 & 6 & 6 & 3 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \to -\frac{1}{2}R_3]{R_1 \to \frac{1}{2}R_1,\ R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in reduced row echelon form

5. Suppose the augmented matrix below is the Reduced Row Echelon Form of an augmented matrix of a linear system. Display the solution of the linear system as a hyperplane (within the space of variables).

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- The equations corresponding to this augmented matrix are

$$\begin{aligned} x_2 - 2x_4 + x_5 &= 1 \\ x_3 + x_4 - x_5 &= 2 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Since columns 1, 4, and 5 do not contain pivots, x_1, x_4 and x_5 should be interpreted as *free variables* in the solution. The above equations then allow us to express x_2 and x_3 in terms of the free variables:

$$\begin{aligned} x_2 &= 1 + 2x_4 - x_5 \\ x_3 &= 2 - x_4 + x_5 \end{aligned}$$

Thus, a typical solution vector would be

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 1 + 2x_4 - x_5 \\ 2 - x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The expression on the right exhibits the solutions as the elements of a hyperplane.

6. Compute the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

- We form the matrix $[\mathbf{A}|\mathbf{I}]$ as

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

Now we row reduce to RREF:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to -R_3]{R_2 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 5 & -2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 3 & 1 \\ 0 & 1 & 0 & 5 & -2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 3 & 1 \\ 0 & 1 & 0 & 5 & -2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] = [\mathbf{I}|\mathbf{A}^{-1}]$$

and so

$$\mathbf{A}^{-1} = \begin{bmatrix} -6 & 3 & 1 \\ 5 & -2 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$