Math 3013
SOLUTIONS TO PRACTICE SECOND EXAM
Spring 2022

1. Define, precisely, the following notions (where V, W are to be regarded as general vector spaces). space V).

(a) (5 pts) a **subspace** of V

- A *subspace* of a vector space V is a subset W of V that is closed under both scalar multiplication and vector addition; i.e.,
  - For all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in W$, $\lambda\mathbf{v} \in W$
  - For all $\mathbf{v}_1, \mathbf{v}_2 \in W$, $\mathbf{v}_1 + \mathbf{v}_2 \in W$

(b) (5 pts) a **basis** for a vector space V

- A *basis* for V is a set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ such that every vector $\mathbf{v} \in V$ can be uniquely expressed as

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k$$

(c) (5 pts) a **set of linearly independent vectors** in V

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is *linearly independent* if the only solution of

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

is $c_1 = 0$, $c_2 = 0$, $\ldots, c_k = 0$

(d) (5 pts) a **linear transformation** from a vector space V to vector space W.

- A *linear transformation* is a function $T : V \to W$ such that
  - $T(\lambda\mathbf{x}) = \lambda T(\mathbf{x})$ for all $\mathbf{x} \in V$
  - $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in V$

2. (10 pts) Prove or disprove that the points on the circle $S = \{[x, y] \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a subspace of $\mathbb{R}^2$.

- A subspace has to be closed under both scalar multiplication and vector addition.
  - *closure under scalar multiplication.* Let $[x, y] \in S$. Then

$$\lambda[x, y] = [\lambda x, \lambda y] \quad \Rightarrow \quad (\lambda x)^2 + (\lambda y)^2 = \lambda^2\left(x^2 + y^2\right) = \lambda^2(1) = \lambda^2 \neq 1$$

So S is not closed under scalar multiplication.

  - *closure under vector addition.* Let $[x_1, y_1], [x_2, y_2] \in S$. Then

$$\begin{aligned} [x_1, y_1] + [x_2, y_2] &= [x_1 + x_2, y_1 + y_2] \\ (x_1 + x_2)^2 + (y_1 + y_2)^2 &= x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 \\ &= \left(x_1^2 + y_1^2\right) + \left(x_2^2 + y_2^2\right) + 2x_1x_2 + 2y_1y_2 \\ &= 2 + 2x_1x_2 + 2y_1y \\ &\neq 1 \text{ in general} \end{aligned}$$

So S is not closed under vector addition.
S is not a subspace

3. (10 pts) Let $W = span([1,1,1],[1,-2,1],[3,0,3]) \subset \mathbb{R}^3$. Find a basis for W.

- $$span(W) = RowSp\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 0 & 3 \end{pmatrix} \xrightarrow{\text{row reduction}} RowSp\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{basis for } W = \text{basis for } RowSp = \{[1,1,1],[0,-3,0]\}$$

4. Consider the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(a) (10 pts) Row reduce this matrix to reduced row echelon form

- $$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (5 pts) Find a basis for the row space of $\mathbf{A}$.

- A basis for the row space of the matrix $\mathbf{A}$ is formed by the non-zero rows of any row echelon form of $\mathbf{A}$. Thus

$$\text{basis for } RowSp(\mathbf{A}) = \{[1,0,0,2],[0,0,1,3]\}$$

(c) (5 pts) Find a basis for the column space of $\mathbf{A}$.

- A basis for the column space of $\mathbf{A}$ is formed by the columns of $\mathbf{A}$ corresponding to the columns of a row echelon form of $\mathbf{A}$ which contain pivots. Since the first and third columns of the RREF of $\mathbf{A}$ are where the pivots of the RREF reside, the first and third columns of $\mathbf{A}$ will be a basis for the column space of $\mathbf{A}$:

$$\text{basis for } ColSp(\mathbf{A}) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(d) (5 pts) Find a basis for the null space of $\mathbf{A}$.

- To find a basis for the null space of $\mathbf{A}$, we must solve $\mathbf{Ax} = \mathbf{0}$. From the reduced row echelon form of $\mathbf{A}$, we conclude that if $\mathbf{x} = [x_1, x_2, x_3, x_4]$ is a solution of $\mathbf{Ax} = \mathbf{0}$, then

$$x_1 = -2x_4$$
$$x_3 = -3x_4$$

and x_2 and x_4 are free parameters: Thus,

$$\mathbf{x} = \begin{bmatrix} -2x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

and

$$\text{basis for } NullSp(\mathbf{A}) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

(e) (5 pts) What is the rank of $\mathbf{A}$?

- $rank(A) = \dim(RowSp(A)) = \dim(ColSp(\mathbf{A})) = 2$

5. (10 pts) Let $\mathbf{A}$ be an $n \times m$ matrix. Show that the solution set of $\mathbf{Ax} = \mathbf{0}$ is a subspace of $\mathbb{R}^m$.

- *closure under scalar multiplication.* Suppose $\mathbf{y}$ is a solution of $\mathbf{Ax} = \mathbf{0}$, then
$$\mathbf{A}(\lambda\mathbf{y}) = \lambda(\mathbf{Ay}) = \lambda\mathbf{0} = \mathbf{0}$$
and $\lambda\mathbf{y}$ is also a solution
- *closure under vector addition.* Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{Ax} = \mathbf{0}$. Then
$$\mathbf{A}(\mathbf{y}_1 + \mathbf{y}_2) = \mathbf{Ay}_1 + \mathbf{Ay}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
and so $\mathbf{y}_1 + \mathbf{y}_2$ is also a solution.
- Since the solution set of $\mathbf{Ax} = \mathbf{0}$ is closed under scalar multiplication and vector addition, it is a subspace.

6. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2 : T([x_1, x_2, x_3, x_4]) = [x_1 - x_3 + 2x_4, 2x_2 - 2x_3 + x_4]$.

(a) (10 pts) Find the matrix $\mathbf{A}_T$ representating T:

- $$\mathbf{A}_T = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ T([1,0,0,0]) & T([1,0,0,0]) & T([1,0,0,0]) & T([1,0,0,0]) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -2 & 1 \end{pmatrix}$$

(b) (5 pts) Find a basis for $range(T) \equiv \{\mathbf{y} \in \mathbb{R}^2 \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^4\}$.

- We have $range(T) = ColSp(\mathbf{A}_T)$. Note that $\mathbf{A}_T$ is already in row echelon form and it has pivots in columns 1 and 2. Therefore, the first two columns of $\mathbf{A}_T$ will be basis vectors. Thus, $\{[1,0], [0,2]\}$ will be a basis for $range(T)$.

(c) (5 pts) Find a basis for $ker(T) \equiv \{\mathbf{x} \in \mathbb{R}^4 \mid T(\mathbf{x}) = \mathbf{0}\}$

- We have $\ker(T) = NullSp(\mathbf{A}_T) =$ solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$. The matrix $\mathbf{A}_T$ is quickly reducible to its reduced row echelon form
$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & -2 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & \frac{1}{2} \end{pmatrix}$$
Fron the RREF of $\mathbf{A}_T$ we see the solutions of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ are vectors of the form
$$\mathbf{x} = \begin{bmatrix} x_3 - 2x_4 \\ x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$
Thus, a basis for $\ker(T)$ is given by
$$\left\{ [1,1,1,0], \left[-2, -\frac{1}{2}, 0, 1\right] \right\}$$