Math 3013

PRACTICE SECOND EXAM

1. Complete the following mathematical definitions

(a) (5 pts) A **subspace** of a vector space V is ...

- a subset W of V such that
  (i) $\lambda \in \mathbb{R}$, $\mathbf{w} \in W \Longrightarrow \lambda \mathbf{w} \in W$
  (ii) $\mathbf{w}_1, \mathbf{w}_2 \in W \Longrightarrow \mathbf{w}_1 + \mathbf{w}_2 \in W$

(b) (5 pts) A **basis** for a subspace W is ...

- a set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ such that
  (i) every vector $\mathbf{w} \in W$ can be expressed as

(*) $$\mathbf{w} = c_1 \mathbf{b}_1 + \cdots + c_k \mathbf{b}_k$$

  (ii) The coefficients $c_1, \ldots, c_k$ in (*) are unique.

(c) (5 pts) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is **linearly independent** if ...

- the only solution of

$$x_1 \mathbf{v}_1 + \cdots + x_k \mathbf{v}_k = \mathbf{0}$$

  is $x_1 = 0, \ldots, x_k = 0$.

(d) (5 pts) A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a **linear transformation** if ...

- (i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.
  (ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

2. Consider the vectors $\{[1,1,1,0],[1,0,1,1],[1,-1,1,2]\} \in \mathbb{R}^4$

(a) (5 pts) Determine if these vectors are linearly independent.

- We'll write these vectors as the rows of a matrix **A** and row reduce to R.E.F.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The zero row at the bottom of the R.E.F. show that the original set of vectors are not linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors?

- The non-zero rows of the R.E.F. will provide a basis for the span of the original set of vectors. Since we have two non-zero rows, the dimension of the subspace generated by the original set of vectors is 2.

3. Given that the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -3 & -1 & -3 \\ 2 & -2 & 0 & -2 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) (5 pts) Find a basis for the row space of **A**.

- A basis for the $RowSp(\mathbf{A})$ is given by the non-zero rows of any R.E.F. of **A**. Thus

$$\text{basis for } RowSp(\mathbf{A}) = \left\{ \left[1, 0, \frac{1}{2}, 0\right], \left[0, 1, \frac{1}{2}, 1\right] \right\}$$

(b) (5 pts) Find a basis for the column space of **A**.

- A basis for $ColSp(\mathbf{A})$ is given by the columns of **A** corresponding to the columns $REF(\mathbf{A})$ that contain pivots. Thus

$$\text{basis for } ColSp(\mathbf{A}) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} \right\}$$

(c) (5 pts) Find a basis for the null space of **A**.

- A basis for $NullSp(\mathbf{A})$ is found by solving the linear system $\mathbf{Ax} = \mathbf{0}$. From the R.R.E.F. of **A** we see that x_3 and x_4 will be free parameters in the solution and

$$\left.\begin{array}{l} x_1 = -\frac{1}{2}x_3 \\ x_2 = -\frac{1}{2}x_3 - x_4 \end{array}\right\} \Rightarrow \mathbf{x} = x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{basis for } NullSp(\mathbf{A}) = \left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) (5 pts) What is the rank of **A**?

- The rank of **A** is the common dimension of $RowSp(\mathbf{A})$ and $ColSp(\mathbf{A})$, which is 2.

4. Consider the following linear transformation: $T : \mathbb{R}^2 \to \mathbb{R}^3 : T([x_1, x_2) = [x_2, x_1 - x_2, x_1 + x_2]$.
(a) (5 pts) Find the matrix $\mathbf{A}_T$ such that $\mathbf{A}_T\mathbf{x} = T(\mathbf{x})$.

- We have

$$T([1,0]) = [0,1,1] \quad , \quad T([0,1]) = [1,-1,1]$$

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow \\ T([1,0]) & T([0,1]) \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(b) (5 pts) Find a basis for the range of T.

$$Range(T) = ColSp(\mathbf{A}_T)$$

- Since $\mathbf{A}_T$ row reduces to

$$R.R.E.F.(\mathbf{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{basis for } Range(T) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) (5 pts) Find the kernel of T.

- $Ker(T) = NullSp(\mathbf{A}_T)$. From the R.R.E.F. of $\mathbf{A}_T$ we see $\mathbf{A}_T\mathbf{x} = \mathbf{0} \iff \mathbf{x} = 0$. And so

$$Ker(T) = \{[0,0]\}$$

5. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

(a) 7 pts) Compute $\det(\mathbf{A})$ via a cofactor expansion along the third column.

$$\begin{aligned} \det(\mathbf{A}) &= a_{13}(-1)^{1+3}\det(\mathbf{M}_{13}) + a_{23}(-1)^{2+3}\det(\mathbf{M}_{23}) \\ &\quad + a_{33}(-1)^{3+3}\det(\mathbf{M}_{33}) + a_{43}(-1)^{4+3}\det(\mathbf{M}_{43}) \end{aligned}$$

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$$\begin{aligned} \det(\mathbf{A}) &= 0 + 0 + (1)(-1)^{3+3}\det\left(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix}\right) + 0 \\ &= 0(-1)^{1+1}\det\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix} + 0(-1)^{2+`}\det\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + (1)(-1)^{3+1}\det\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \\ &= (1)(1)((2)(3) - (0)(1)) = 6 \end{aligned}$$

(b) (8 pts) Compute $\det(\mathbf{A})$ by row reducing $\mathbf{A}$ to an upper triangular matrix.

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$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

We have

$$\det(\mathbf{A}) = (-1)^1\det(R.E.F.(\mathbf{A})) = (-1)(1)(1)(1)(-6) = 6$$

since we did 1 row interchange and 0 row rescalings in our our row reduction, and the determinant of the R.E.F. is just the product of its diagonal entries.

6. (10 pts) Use Cramer's Rule to solve

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ x_1 - x_2 &= 3 \end{aligned}$$

- We have

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

and

$$\begin{aligned} \det(\mathbf{A}) &= (2)(-1) - (1)(1) = -3 \\ \det(\mathbf{B}_1) &= \det\left(\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}\right) = (3)(-1) - (1)(3) = -6 \\ \det(\mathbf{B}_2) &= \det\left(\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}\right) = (2)(3) - (3)(1) = 3 \end{aligned}$$

Hence, by Cramer's Rule,

$$\begin{aligned} x_1 &= \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{-6}{-3} = 2 \\ x_2 &= \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = \frac{3}{-3} = -1 \end{aligned}$$

7. (10 pts) Find all the cofactors of $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$, and then use these cofactors to compute $\mathbf{A}^{-1}$

- The cofactors are

$$c_{ij} = (-1)^{i+j} \det(\mathbf{M}_{ij})$$

where $\mathbf{M}_{ij}$ is the 1×1 matrix obtained from $\mathbf{A}$ by deleting the i^{th} row and j^{th} column of $\mathbf{A}$. We have

$$\begin{aligned} c_{11} &= (-1)^{1+1} \det([1]) = 1 \quad , \quad c_{12} = (-1)^{1+2} \det([1]) = -1 \\ c_{21} &= (-1)^{2+1} \det([3]) = -3 \quad , \quad c_{22} = (-1)^{2+2} \det([2]) = 2 \end{aligned}$$

and so

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{C}^T = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} \\ &\Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \frac{1}{(2)(1) - (3)(1)} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \end{aligned}$$