Math 3013
Solutions to Sample Final Exam

1. Give the definitions of the following linear algebraic concepts:

(a) (5 pts) a **subspace** of a vector space $V$.

- A subspace of a vector space $V$ is a subset $W$ of $V$ that is closed under scalar multiplication and vector addition: i.e.,

$$\lambda \in \mathbb{R}\ ,\ \mathbf{w} \in W \quad \Rightarrow \quad (\lambda \mathbf{w}) \in W$$
$$\mathbf{w}, \mathbf{u} \in W \quad \Rightarrow \quad (\mathbf{w} + \mathbf{u}) \in W$$

(b) (5 pts) a **basis** for a subspace $W$ of a vector space $V$

- A basis for a subspace $W$ of a vector space $V$ is a set of vectors $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ such that each vector $\mathbf{w} \in W$ has a unique expresson as a linear combination of the vectors in $B$: i.e., each $\mathbf{w} \in W$ can be expressed as

$$\mathbf{w} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k$$

in one and only one way.

(c) (5 pts) a set of **linearly independent** vectors

- A set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of vectors is linearly independent if the only solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ is $c_1 = 0\ ,\ c_2 = 0\ ,\ \dots\ ,\ c_k = 0$

(d) (5 pts) a **linear transformation** between two vector spaces $V$ and $W$.

- A linear transformation from a vector space $V$ to a vector space $W$ is a function $T : V \to W$ such that
  - $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ for all $\mathbf{v} \in V$ and all $\lambda \in \mathbb{R}$.
  - $T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}')$ for all $\mathbf{v}, \mathbf{v}' \in V$.

2. For each of the following augmented matrices, describe the solution space of the corresponding linear system. (Determine if there are solutions; and, if there are solutions, how many free parameters are needed to describe the general solution.)

(a) (5 pts) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- This augmented matrix corresponds to a system of 4 linear equations in four unknowns. It has a solution since there are no pivots in the last column of the augmented matrix. There two columns without pivots in this row echelon form and so the solution set is 2-dimensional

(b) (5 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- This equation corresponding to the third row implies $0 = 1$, and so there is no solution

(c) (5 pts) $\left[\begin{array}{ccccc|c} 0 & -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

This augmented matrix corresponds to a linear system with 4 equations in 5 unknowns. It does have a solution. There are two columns without pivots in this row echelon form matrix and so the solution space is 2-dimensional.

3. (10 pts) Compute the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{bmatrix}$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

4. (10 pts) Let $W = \{[x, y, z] \in \mathbb{R}^2 \mid x + y + z = 1 \in \mathbb{R}\}$. Prove or disprove that $W$ is a subspace of $\mathbb{R}^2$.

5. Consider the vectors $\{[1, -2, 2, 1], [1, -1, 3, 1], [0, 1, 1, 0], [2, -3, 5, 2]\} \in \mathbb{R}^4$

(a) (10 pts) Determine if these vectors are linearly independent.

- We write the vectors as the rows of a matrix and row reduced to row echelon form

$$\begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & -3 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  The two non-zero roows in the row echelon form indicate that the original set of vectors were not linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors (i.e. the subspace spanned by these vectors)?

- The non-zero rows of the row echelon form form a basis for the row space. Since there are two basis vectors, the subspace generated by the original set of vectors is 2-dimensional.

6. Consider the following linear transformation: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2.x_3]) = [x_2 - x_3\ ,\ x_1 - x_3]$.
(a) (10 pts) Find a matrix that represents $T$.

- We have

$$\begin{aligned} T([1,0,0]) &= [0,1] \\ T([0,1,0]) &= [1,0] \\ T([0,0,1]) &= [-1,-1] \end{aligned}$$

To get the matrix $\mathbf{A}_T$ we write the vectors on the right as columns:

$$\mathbf{A}_T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

(b) (10 pts) Find a basis for the kernel of $T$ (i.e. the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$).

- We have

$$\ker(T) = NullSp(\mathbf{A}_T) = NullSp\left(\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\right)$$
$$= span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

Thus,

$$\ker(T) = span([1,1,1])$$

7. (a) (15 pts) Find the eigenvalues and the eigenvectors of the following matrix : $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

- Carrying out a cofactor expansion alon the bottom row we get

$$p_{\mathbf{A}}(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0+0+(-1)^{3+3}(1-\lambda)\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = -\lambda(1-\lambda)^2$$

and so the eigenvalues are $\lambda = 0, 1$.
$\lambda = 0$ eigenspace:

$$NullSp(\mathbf{A} - (0)\mathbf{I}) = NullSp\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = NullSp\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$
$$\Rightarrow \quad \mathbf{v}_{\lambda=0} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 1$ eigenspace:

$$NullSp(\mathbf{A} - (0)\mathbf{I}) = NullSp\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = NullSp\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \quad \mathbf{v}_{\lambda=1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

(b) (5 pts)Is this matrix diagonalizable?

- No. In part (a) we only found 2 linearly independent eigenvectors. We need three linearly independent eigenvectors to diagonalize a $3 \times 3$ matrix.

8. (15 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$, Find a $2\times 2$ matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{AC} = \mathbf{D}$.

- We first need to find the eigenvalues and eigenvectors of $\mathbf{A}$.

$$0 = p_{\mathbf{A}}(\lambda) = \det\begin{pmatrix} -1-\lambda & 5 \\ 1 & 3-\lambda \end{pmatrix} = (-1-\lambda)(3-\lambda) - 5 = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2)$$

$\Rightarrow \quad \lambda = 4, 2$

– $\lambda = 4$ eigenspace:

$$NullSp(\mathbf{A} - 4\mathbf{I}) = NullSp\begin{pmatrix} -5 & 5 \\ 1 & -1 \end{pmatrix} = NullSp\begin{pmatrix} 1 & -1 \\ 0 & - \end{pmatrix} = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$\Rightarrow \quad \mathbf{v}_{\lambda=4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -2$ eigenspace:

$$NullSp(\mathbf{A} - (-2)\mathbf{I}) = NullSp\begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 5 \\ 0 & 0 \end{pmatrix} = span\begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{v}_{\lambda=-2} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

Since we have two linearly independent eigenvectors we can use them to construct an invertible matrix $\mathbf{C}$ that diagonalizes $\mathbf{A}$, and use their eigenvalues to construct the diagonal matrix $\mathbf{D}$:

$$\mathbf{C} = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

9. (15 pts)Let $\mathbf{v} = [1,1,1]$ and let $W = span([1\ ,1,0]\,,\ [0,1,1])$. Find the orthogonal decomposition $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$ of $\mathbf{v}$ with respect to the subspace $W$.

- We are given a basis for $W$, and so first we need to find a basis for $W^\perp = \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v}\cdot\mathbf{w} = \mathbf{0}$ for all $\mathbf{v}$ This is done by writing the generators of $W$ as the rows of a matrix and then finding a basis for the null space of this matrix.

$$NullSp\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = span\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$$

Thus,

$$W^\perp = span[1,-1,1]$$

Adjoining the basis for $W^\perp$ to bhe basis for $W$, gives us a basis for $\mathbb{R}^3$

$$B = \{[1,1,0]\,,[0,1,1]\,,[1,-1,1]\}$$

We'll now figure out the coordinate vector of $\mathbf{v} = [1,1,1]$ with respect to the basis $B$. The condition

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_2$$

amounts to solving the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which in turn can be solved using augmented matrices and row reduction

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

Thus,

$$\mathbf{v} = \left(\frac{2}{3}\right)[1,1,0] + \frac{2}{3}[0,1,1] + \left(\frac{1}{3}\right)[1,-1,1]$$

The sum of the first two vectors is the component $\mathbf{v}_W$ of $\mathbf{v}$ lying in $W$ and the last vector is the component $\mathbf{v}_\perp$ of $\mathbf{v}$ lying in $W^\perp$. Thus,

$$\mathbf{v}_w = \left[\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right]$$
$$\mathbf{v}_\perp = \left[\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right]$$

10. (10 pts) Find an orthonormal basis for the subspace $W$ generated by the vectors $\mathbf{v}_1 = [1,1,1]$ and $\mathbf{v}_2 = [1,0,1]$

- We'll first find an orthogonal basis $\{\mathbf{o}_1, \mathbf{o}_2\}$ for $W$. We first

$$\mathbf{o}_1 = \mathbf{v}_1 = [1,1,1]$$

and then set $\mathbf{o}_2$ to be the component of $\mathbf{v}_2$ that is perpendicular to $\mathbf{o}_1$ :

$$\begin{aligned} \mathbf{o}_2 &= \mathbf{v}_2 - \frac{\mathbf{o}_1 \cdot \mathbf{v}_2}{\mathbf{o}_1 \cdot \mathbf{o}_1}\mathbf{o}_1 \\ &= [1,0,1] - \left(\frac{2}{3}\right)[1,1,1] \\ &= \left[\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right] \end{aligned}$$

Note $\{\mathbf{o}_1, \mathbf{o}_2\}$ are linearly independent, span $W$ and $\mathbf{o}_1 \cdot \mathbf{o}_2 = 0$ and so this is an **orthogonal** basis for $W$. To obtain an **orthonormal** basis, we need to renormalize $\mathbf{o}_1$ and $\mathbf{o}_2$ so that they have unit length.

$$\begin{aligned} \|\mathbf{o}_1\| &= \sqrt{\mathbf{o}_1 \cdot \mathbf{o}_1} = \sqrt{3} \\ \|\mathbf{o}_2\| &= \sqrt{\mathbf{o}_2 \cdot \mathbf{o}_2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}} = \frac{\sqrt{6}}{3} \end{aligned}$$

And so the vectors

$$\begin{aligned} \mathbf{n}_1 &= \frac{\mathbf{o}_1}{\|\mathbf{o}_1\|} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \\ \mathbf{n}_2 &= \frac{\mathbf{o}_2}{\|\mathbf{o}_2\|} = \left[\frac{3}{\sqrt{6}}, -\frac{6}{\sqrt{6}}, \frac{3}{\sqrt{6}}\right] \end{aligned}$$

will be an orthonormal basis for $W$