Math 3013
SOLUTIONS TO SECOND PRACTICE EXAM

1. Consider the vectors $\{[1,1,1,0],[1,0,1,1],[1,-1,1,2],[1,0,0,-1]\} \in \mathbb{R}^4$

(a) (10 pts) Determine if these vectors are linearly independent.

- We form a matrix using the given vectors as rows and row reduce the matrix.

$$\mathbf{A} = \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 1 & 0 & 0 & -1 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathbf{A}'$$

  The zero row at the bottom of the reduced row echelon form $\mathbf{A}'$ indicates a dependency relation amongst the original set of vectors. Therefore, the verctors are not linearly independent.

(b) (5 pts) What is the dimension of the subspace generated by these vectors?

- The nonzero rows of the row echelon form $\mathbf{A}'$ provide a basis for the span of the original set of vectors. Since there are three basis vectors the dimension of this subspace is 3.

2. Write the definitions (as stated in class) of the following notions. (5 pts each)

(a) (A subset of $\mathbb{R}^n$ that is) **closed under scalar multiplication**

- $W \subset \mathbb{R}^n$ is **closed under scalar multiplication** if whenever $\lambda \in \mathbb{R}$ and $\mathbf{w} \in W$, $\lambda\mathbf{w} \in W$.

(b) (A subset of $\mathbb{R}^n$ that is) **closed under vector addition.**

- $W \subset \mathbb{R}^n$ is **closed under vector addition** if whenever $\mathbf{w}, \mathbf{v} \in W$, $\mathbf{w} + \mathbf{v} \in W$.

(c) A **subspace** of $\mathbb{R}^n$.

- A **subspace** of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that is closed under scalar multiplication and vector addition

(d) A **basis** for a subspace of $\mathbb{R}^n$.

- A set B of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ is a **basis** for a subspace W of $\mathbb{R}^n$ if every vector $\mathbf{w}$ in W can be uniquely expressed as a linear combination of the vectors in B: for each $\mathbf{w} \in W$

$$\mathbf{w} = c_2\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k \text{ for one and only one choice of scalars } c_1, c_2, \ldots, c_k$$

(e) A set of **linearly independent vectors**

- A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors is **linearly independent** if the only solution of

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

  is

$$c_1 = 0 \ , \ c_2 = 0 \ , \ \ldots \ , \ c_{k=0}$$

3. Given that the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -3 & -1 & -3 \\ 2 & -2 & 0 & -2 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) (5 pts) Find a basis for the row space of $\mathbf{A}$.

- The non-zero rows of the row echelon form provide a basis for the row space of $\mathbf{A}$:

$$\text{basis for } RowSp(\mathbf{A}) = \left\{ \left[1, 0, \frac{1}{2}, 0\right], \left[0, 1, \frac{1}{2}, 1\right] \right\}$$

(b) (5 pts) Find a basis for the column space of $\mathbf{A}$.

- The columns of $\mathbf{A}$ corresponding to the columns of the row echelon form that contain pivots will provide a basis for the column space of $\mathbf{A}$:

$$\text{basis for } ColSp(\mathbf{A}) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ -2 \end{bmatrix} \right\}$$

(c) (5 pts) Find a basis for the null space of $\mathbf{A}$.

- The null space of $\mathbf{A}$ is the solution set of $\mathbf{Ax} = \mathbf{0}$. The row echelon form of $\mathbf{A}$ given above is actually in reduced row echelon form, and from this matrix we can read off the equations of the solutions:

$$\left.\begin{array}{r} x_1 + \frac{1}{2}x_3 = 0 \\ x_2 + \frac{1}{2}x_3 + x_4 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right\} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_3 \\ -\frac{1}{2}x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\text{basis for } NullSp(\mathbf{A}) = \left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) (5 pts) What is the rank of $\mathbf{A}$?

- $rank(\mathbf{A}) = \dim RowSp(\mathbf{A}) = \dim ColSp(\mathbf{A}) = \#$ pivots in row echelon form of $\mathbf{A} = 2$.

4. Consider the following linear transformation: $T : \mathbb{R}^2 \to \mathbb{R}^4 : T([x_1, x_2) = [x_2, x_1, x_1 - x_2, x_1 + x_2]$.

(a) (10 pts) Find a matrix that represents T.

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$$\begin{array}{l} T([1,0]) = [0,1,1,1] \\ T([0,1]) = [1,0,-1,1] \end{array} \quad \Rightarrow \quad \mathbf{A}_T = \begin{bmatrix} | & | \\ T([1,0]) & T([0,1]) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(b) (5 pts) Find a basis for the range of T.

- $Range(T) = ColSp(\mathbf{A}_T)$. The $\mathbf{A}_T$ row reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since each column of the row echelon form contains a pivot, each column of $\mathbf{A}_T$ is a basis vector for the column space of $\mathbf{A}_T$, and hece a basis vector for the range of T :

$$\text{basis for range of } T = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) (5 pts) Find a basis for the kernel of T.

- The kernel of T is equal to the null space of $\mathbf{A}_T$. From the reduced row echelon form of $\mathbf{A}_T$ we see that the only solution of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = 0$. Thus,

$$\ker(T) = \{\mathbf{0}\}$$

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5. Compute the determinants of the following matrices (5 pts each)

(a) $\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = 0$ since the rows are not linearly independent

(b) $\det \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 0 + 0 + 0 + (-1)^{4+4} (3) \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$= (3) \left(0 + (-1)^{2+2} (2) \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 0 \right)$

$= (3)(2)(1 - 0) = 6$

(c) $\det \begin{bmatrix} 1 & 1 & 3 & 1 & 0 \\ 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ because the row operation $R_2 \to R_2 - R_1$ does not change the determinant. The resulting matrix is upper triangular, so its determinant is just the product of its diagonal entries. Thus, the determinant will be

$$\det = (1)(1)(3)(2)(1) = 6$$