# Lecture 1: Introduction to Math 3013

Linear Algebra

Math 3013

Oklahoma State University

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# Agenda

- Course Setup
  - Syllabus
  - Canvas Demo
  - WebAssign Demo
- Lecture 1
  - Why Are We Here?
  - How to Succeed in Math 3013
  - Vectors and Vector Spaces
  - Sets and Notation for Sets
  - Two Fundamemental Vector Operations
  - Vector Operations in Geometry and Physics

# Why are we here?

- Linear Algebra in the Physical Sciences
  - Vectors are used to represent physical quantities with both a direction and magnitude
    - Examples: Position variables , Velocities , Accelerations , Forces , Torques , Physical Fields (Electric, Gravitational, . . . ) , :.
- Linear Algebraic Formalism
  - provides a calculational platform for working with vectors
  - provides a universal methodology for working with things that somehow behave like vectors
    - vibrations and oscillations
    - solution sets of systems of linear equations
    - function spaces ; e.g. solution sets of differential equations
  - provides a **language** for understanding physical laws governing vector-like phenomena

# How to succeed in Math 3013

**Calculations**

- Warning: Calculations tend to be rote, and tedious, in linear algebra
- But explicit calculations are needed to provide concrete examples that illustrate the theory
- Do all homework computations by hand
  "learning through your fingers" is best way to get a handle on the underlying theory
- Take care to write your mathematical work neatly
  - this will help you avoid errors in calculations
  - this will also keep your thinking tidy
- "The devil is in the detail" should be your mantra for this course. Once you start glossing over details, you'll lose the focus you'll need to compute accurately in this course.
- In short, **strive to be neat, patient, and careful as you calculate**.

# How to succeed in Math 3013, cont'd

**Understanding the Theory**

- This is not a proof course
- Nevertheless, the fundamental ideas will be introduced formally and developed logically
- In fact, everything in the course is derived from just a few basic definitions
- Memorize the basic definitions
  - slipshod understandings of the fundamental definitions will be your nemesis in this course
  - conceptual precision is essential
  - it is best to memorize fundamental definitions **verbatim**
  - be aware that a deeper understanding of the material will come only as you begin apply the definitions you've memorized
- Be disciplined in surmounting vague understandings of things.
- **Always feel free to ask questions or request examples**

# What are vectors?

**Vectors in Geometry**

- a geometric vector is a line segment of a particular length heading in a particular direction

Figure: A geometric vector

# What are vectors, cont'd

**Vectors in Physics and Engineering**

- a **physical vector** is a physical quantity with both a magnitude and direction Examples: coordinate vectors, velocities, accelerations, forces, torques, etc.
- Some phenomena, like vibrational modes, or the states of an atom or elementary particle also representable as vectors (typically in an infinite dimensional vector space)

# Vectors in Math 3013

## Definition

A (real) **vector** in Math 3013 is an ordered list of real numbers.

- particular vectors are specified by writing the elements in order and enclosing them in square brackets. Thus,

$$[-1, 5, \sqrt{2}]$$

  is a vector in the context of Math 3013.
- The number of real numbers in a vector is its **dimension**
- the individual numbers are referred to as the **components** of the vector.
  E.g., $[-1, 5, \sqrt{2}]$ is a 3-dimensional vector with components $-1$, 5, and $\sqrt{2}$.
- Two vectors are equal to one another if and only if both the numbers in the lists and the ordering of these numbers is the same. Thus,

$$[-1, 5, \sqrt{2}] \neq [-1, \sqrt{2}, 5]$$

# Sets and Notation for Sets

- A **set** is collection of objects – mathematical objects in this course.
- The objects of a set are sometimes referred to as the **elements** of the set. We indicate that a particular object $s$ is an element of a set $S$ using the symbol $\in$. Thus,

$$s \in S \qquad \Longleftrightarrow \qquad s \text{ is an element of } S$$

- A **subset** of a set $S$ is a set $T$ such that whenever an element $t$ belongs to $T$ then $t$ belongs to $S$. This is indicated using the symbol $\subset$ :

$$T \subset S \qquad \Longleftrightarrow \qquad T \text{ is a subset of } S$$

# Sets cont'd

- The **empty set** is the set containing no elements. It is denoted by $\{\}$. It is a subset of every set.

$$\text{If } S \text{ is a set} \qquad \Longrightarrow \qquad \{\} \subset S$$

- Every set is also a subset of itself.
- Thus, every set has at least two subsets: itself and $\{\}$ (well, except the empty set where the empty set coincides with itself)

# Specifying Sets

Two ways of denoting particular sets

- Writing each element down, separated by commas, and enclosed inside curly braces:

  $\{1, 5, 7\}$ is the set whose elements are the numbers 1, 5 and 7

- Specifying a set by stating a property or characteristic that distinguishes the elements of that set E.g.,

  $S = \{x \mid x \text{ is an even integer } \} \Longrightarrow S$ is the set of all even integers

  In the notation above, the symbol "$|$" should be read as the phrase "such that"

# $\mathbb{R}$ and $\mathbb{R}^n$

There are two fundamental sets we will use all the time in Math 3013 and so they'll have their own symbols.

- $\mathbb{R}$ : the set of all real numbers

$$\mathbb{R} \equiv \{x \mid x \text{ is a real number } \}$$

- $\mathbb{R}^n$ : the set of all $n$-dimensional vectors

$$\mathbb{R}^n \equiv \{[x_1, x_2, \ldots, x_n] \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

In the notation above, the ellipses "..." are used to indicate that the pattern begun is to continue. Thus, above we are saying $\mathbb{R}^n$ is the set whose elements are ordered lists of $n$ objects $x_1, x_2, \ldots, x_n$ such that each of these objects is a real number

# Remarks

- To show that two sets $S$ and $T$ are equal, one has to demonstrate two things
  (i) $s \in S \implies s \in T$
  (ii) $t \in T \implies t \in S$

  Put another way,

$$S = T \iff S \subset T \text{ and } T \subset S$$

- Unlike ordered lists (such as vectors), the ordering of the elements of a set is immaterial

$$\{1, 3, 7\} = \{7, 1, 3\} \text{ even though } [1, 3, 7] \neq [7, 1, 3]$$

# The Fundamental Vector Operations

Recall that, in Math 3013, vectors are just ordered lists of numbers and that

$$\mathbb{R}^n \equiv \{[x_1, x_2, \ldots, x_n] \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

is the set of all $n$-dimensional vectors.
We will now introduce two simple, but fundamental operations, we can apply to vectors to get other vectors.

- Vector Addition : $+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
- Scalar Multiplication: $* : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Here, if $A$ and $B$ are sets, $A \times B$ is the **Cartesian product** of $A$ and $B$:

$$A \times B \equiv \{(a, b) \mid a \in A, b \in B\}$$

# Vector Addition

### Definition

Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ be two vectors in $\mathbb{R}^n$. The vector sum of $\mathbf{x}$ and $\mathbf{y}$ is defined as follows:

$$\mathbf{x} + \mathbf{y} = [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$$

In other words, one adds vectors component-by-component. Here is a simple vector addition computation:

$$[1, 3, -1] + [2, 0, 1] = [1 + 2, 3 + 0, -1 + 1] = [3, 3, 0]$$

# Vector Operations, cont'd

## Definition

Let $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ and let $\lambda \in \mathbb{R}$. The **scalar multiple** of $\mathbf{x}$ by $\lambda$ is denoted $\lambda\mathbf{x}$ and is defined as follows

$$\lambda\mathbf{x} \equiv [\lambda x_1, \lambda x_2, \ldots, \lambda x_n]$$

Thus, for example, if $\mathbf{x} = [1, 3, 2]$ and $\lambda = -2$ , then

$$\lambda\mathbf{x} = [(-2)(1), (-2)(3), (-2)(2)] = [-2, -6, -4]$$

# Vector Operations in Plane Geometry

- In plane geometry, geometric vectors are represented as directed line segments, or arrows, drawn on a piece of paper.
- Such vectors can be related to 2-dimensional real vectors as follows:
  - Choose a Cartesian coordinate system such that the base of the directed line segment is at the origin.
  - The coordinates of the tip of the vector will be an ordered list of two real numbers. That ordered list of numbers is the linear algebraic vector corresponding to the original geometric vector.

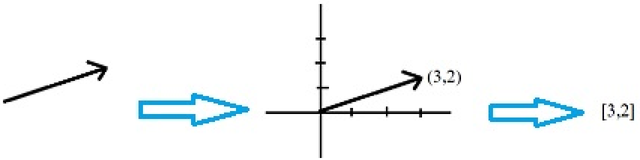


Figure: geometric vector to linear algebraic vector

# Vector Operations in Plane Geometry, cont'd

We can also go in the opposite way. To find the geometric vector corresponding to the linear algebraic vector $[2, 3]$, we simply

- choose an origin on a piece of paper
- draw an arrow whose base is at the origin and whose tip is has coordinates $[2, 3]$

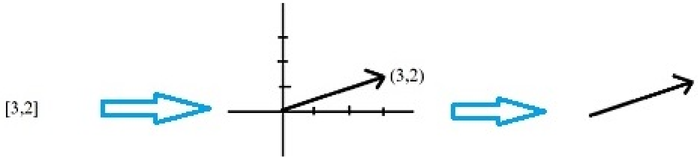


Figure: linear algebraic vector to geometric vector

# Vector Addition for Geometric Vectors

One adds geometric vectors as follows.

- transport (without changing its length or direction) the second vector so that its base is at the tip of the first vector
- draw a new arrow from the base of the first vector to the tip of the second vector.

This newly drawn arrow is the geometric vector corresponding the sum of the original two vectors.

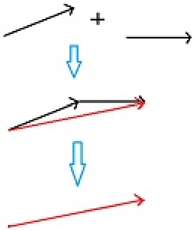


Figure: adding geometric vectors

# Scalar Multiplication of Geometric Vectors

Scalar multiplication of a geometric vector by a positive number $\lambda$ just rescales its length by a factor of $\lambda$.

Scalar multiplication of a geometric vector by a negative number $\lambda$ not only rescales the length of the vector the (positive) factor $\mid \lambda \mid$, it also reverses its direction.

To demonstrate this, let's consider the scalar multiplication of the geometric vector corresponding to the linear algebraic vector $[2, 1]$ by $-2$. Linear algebraically we have $(-2) * [2, 1] = [-4, 2]$. So the corresponding planar vector will be

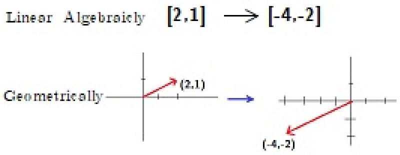


Figure: scalar multiplying vectors

# Vectors in Plane Geometry vs. Vectors in Linear Algebra

This course is concerned primarily with vectors represented as ordered lists of numbers.

- We don't really have the ability to work with geometric vectors if they live in spaces of dimension greater than 3 (and even 3-dimensional geometric vectors are awkward to work with).
- We can easily work with ordered lists of numbers, no matter what the dimension.
- However, geometric vectors in 2-dimensions do provide us a way of visualizing what we are doing as we manipulate vectors in linear algebra.

# Vector Operations in Physics and Engineering

- When forces are represented as vectors, adding them is interpreted as combining forces (to get the total force acting on an object).
- When vibrations, like sound waves, are represented as vectors
  - adding vectors corresponds to combining sounds (like the way individual instruments produce the sound of an orchestra).
  - scalar multiplying vectors by a positive number $\lambda$ corresponds to increasing/decreasing the amplitude of the corresponding vibration by a factor of $\lambda$
  - scalar multiplication by a negative number $\lambda$ corresponds to changing the amplitude of the vibration by a factor $\lambda$ and reversing the *phase* of the vibration (this is the way noise cancelling headphones work)

# Linear Combinations

The operations of vector and addition are readily composed with one another.
For example, if $\lambda, \nu \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then we can scalar multiply $\mathbf{a}$ by $\lambda$, and $\mathbf{b}$ by $\nu$, and then add the two resulting vectors. We can then indicate the resulting vector as

$$\lambda \mathbf{a} + \nu \mathbf{b}$$

More generally,

## Definition

A **linear combination** of $n$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a vector of the form

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n$$

for some choice of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

# Vector Subtraction

A particularly useful linear combination of two vectors **a** and **b** is their vector difference

$$\mathbf{a} - \mathbf{b} \equiv \mathbf{a} + (-1)\mathbf{b}$$

Vector subtraction is used to indicate the *displacement* between two vectors. When the vectors $\mathbf{a}, \mathbf{b}$ are represented geometrically as directed line segments, $\mathbf{a} - \mathbf{b}$ corresponds to the directed line segment that goes from the tip of **b** to the tip of **a**.