Math 3013 Lecture 2

January 12, 2022

Agenda

- Summary of First Lecture
- Vectors in $\mathbb{R}^n$ and the Two Fundamental Vector Operations

Sets vs. Ordered Lists

A **set** is simply a collection of mathematical objects.
We use curly braces to denote sets:

$$\left\{1, -3, \sqrt{2}\right\} = \text{ the set containing the numbers } 1, -3, \text{ and } \sqrt{2}$$

An **ordered list** is a collection of mathematical objects written in a particular order.
We use parentheses or square braces to indicate ordered lists

$$(a, c, b) \quad \text{or} \quad [a, c, b]$$

Vectors in Math 3013

Definition

A (real) **vector** in Math 3013 is an ordered list of real numbers.

- particular vectors are specified by writing the elements in order and enclosing them in square brackets (which is the convention for writing ordered lists of numbers) . Thus,

  $$[-1, 5, \sqrt{2}]$$

  is a vector in the context of Math 3013.
- The number of real numbers in a vector is its **dimension**
- the individual numbers are referred to as the **components** of the vector.
  E.g., $[-1, 5, \sqrt{2}]$ is a 3-dimensional vector with components -1, 5, and $\sqrt{2}$.
- Two vectors are equal to one another if and only if both the numbers in the lists and the ordering of these numbers is the same. Thus,

  $$[-1, 5, \sqrt{2}] \neq [-1, \sqrt{2}, 5]$$

Specifying Sets

Two standard ways to specify particular sets

- Writing each element down, separated by commas, and enclosed inside curly braces:

$$S \equiv \{\sqrt{2}, -\sqrt{2}\}$$

- Specifying a set by stating a property or characteristic that distinguishes the elements of that set. E.g.,

$$S \equiv \{x \in \mathbb{R} \mid x^2 = 2\}$$

(Here the symbol $\equiv$ is used to indicate a mathematical definition).
In the notation above, the symbol "$|$" should be read as the phrase "such that"
Note that

$$\{x \in \mathbb{R}^2 \mid x^2 = 2\} = \{\sqrt{2}, -\sqrt{2}\}$$

Equality of Sets

To show that two sets S and T are the same set, one has to demonstrate two things

(i) $s \in S \implies s \in T$

(ii) $t \in T \implies t \in S$

Put another way,

$$S = T \iff S \subset T \text{ and } T \subset S$$

Cartesian Product of Sets

Definition/Notation: Suppose A and B are sets. The **Cartesian product** of A and B denoted $A \times B$ and it is the set of all ordered pairs (a, b) where a is an element of A and b is an element of B:
In set notation, this definition is

$$A \times B \equiv \{(a, b) \mid a \in A, b \in B\}$$

$\mathbb{R}$ and $\mathbb{R}^n$

There are two fundamental sets we will use all the time in Math 3013 and so they'll have their own symbols.

- $\mathbb{R}$: the set of all real numbers

$$\mathbb{R} \equiv \{x \mid x \text{ is a real number }\}$$

- $\mathbb{R}^n$: the set of all n-dimensional vectors

$$\mathbb{R}^n \equiv \{[x_1, \ldots, x_n] \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

In the notation above, the ellipses "$\ldots$" are used to indicate that the pattern begun is to continue. Thus, above we are saying $\mathbb{R}^n$ is the set whose elements are ordered lists of n objects $x_1, x_2, \ldots, x_n$ such that each of these objects is a real number

Remark: Note how our notation for $\mathbb{R}^n$ mimics its construction via Cartesian products of $\mathbb{R}$ with itself. E.g.,

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

Back to Linear Algebra: The Fundamental Vector Operations

Recall that, in Math 3013, vectors are just ordered lists of numbers and that

$$\mathbb{R}^n \equiv \{[x_1, x_2, \ldots, x_n] \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

is the set of all *n*-dimensional vectors.

We will now introduce two simple, but fundamental operations, we can apply to vectors to get other vectors.

- Vector Addition : $+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
- Scalar Multiplication: $* : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Vector Addition

Definition

Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ be two vectors in $\mathbb{R}^n$. The **vector sum** of $\mathbf{x}$ and $\mathbf{y}$ is defined as follows:

$$\mathbf{x} + \mathbf{y} = [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$$

In other words, one adds vectors component-by-component. Here is a simple vector addition computation:

$$[1, 3, -1] + [2, 0, 1] = [1 + 2, 3 + 0, -1 + 1] = [3, 3, 0]$$

Vector Operations, cont'd

Definition

Let $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ and let $\lambda \in \mathbb{R}$. The **scalar multiple** of $\mathbf{x}$ by λ is denoted $\lambda\mathbf{x}$ and is defined as follows

$$\lambda\mathbf{x} \equiv [\lambda x_1, \lambda x_2, \ldots, \lambda x_n]$$

Thus, for example, if $\mathbf{x} = [1, 3, 2]$ and $\lambda = -2$, then

$$\lambda\mathbf{x} = [(-2)(1), (-2)(3), (-2)(2)] = [-2, -6, -4]$$

Vector Operations in Plane Geometry

- In plane geometry, geometric vectors are represented as directed line segments, or arrows, drawn on a piece of paper.
- Such vectors can be related to 2-dimensional real vectors as follows:
  - Choose a Cartesian coordinate system such that the base of the directed line segment is at the origin.
  - The coordinates of the tip of the vector will be an ordered list of two real numbers. That ordered list of numbers is the linear algebraic vector corresponding to the original geometric vector.

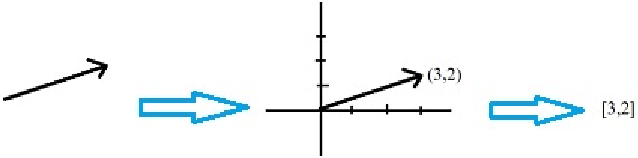


Figure: geometric vector to linear algebraic vector

Vector Operations in Plane Geometry, cont'd

We can also go in the opposite way. To find the geometric vector corresponding to the linear algebraic vector $[2, 3]$, we simply

- choose an origin on a piece of paper
- draw an arrow whose base is at the origin and whose tip is has coordinates $[2, 3]$

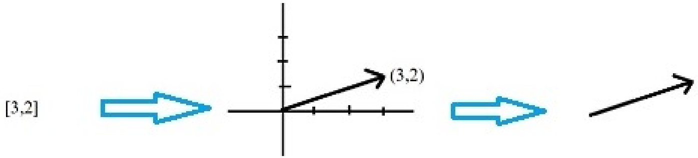


Figure: linear algebraic vector to geometric vector

Vector Addition for Geometric Vectors

One adds geometric vectors as follows.

- transport (without changing its length or direction) the second vector so that its base is at the tip of the first vector
- draw a new arrow from the base of the first vector to the tip of the second vector.

This newly drawn arrow is the geometric vector corresponding the sum of the original two vectors.

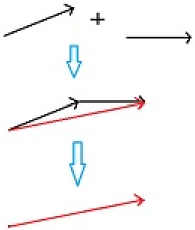


Figure: adding geometric vectors

Scalar Multiplication of Geometric Vectors

Scalar multiplication of a geometric vector by a positive number λ just rescales its length by a factor of λ.

Scalar multiplication of a geometric vector by a negative number λ not only rescales the length of the vector the (positive) factor $\mid \lambda \mid$, it also reverses its direction.

To demonstrate this, let's consider the scalar multiplication of the geometric vector corresponding to the linear algebraic vector $[2, 1]$ by -2. Linear algebraically we have $(-2) * [2, 1] = [-4, 2]$. So the corresponding planar vector will be

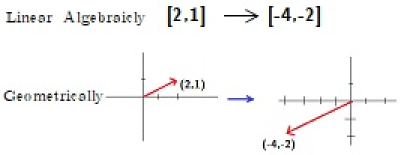


Figure: scalar multiplying vectors

Vectors in Plane Geometry vs. Vectors in Linear Algebra

This course is concerned primarily with vectors represented as ordered lists of numbers.

- We don't really have the ability to work with geometric vectors if they live in spaces of dimension greater than 3 (and even 3-dimensional geometric vectors are awkward to work with).
- We can easily work with ordered lists of numbers, no matter what the dimension.
- However, geometric vectors in 2-dimensions do provide us a way of visualizing what we are doing as we manipulate vectors in linear algebra.

Vector Operations in Physics and Engineering

- When forces are represented as vectors, adding them is interpreted as combining forces (to get the total force acting on an object). Scalar multiplication is used, for example, to change units of measurements.
- In acoustics, air pressure fluctuations, i.e., sound waves, can be represented as vectors
  - adding vectors corresponds to combining sounds (like the way individual instruments produce the sound of an orchestra).
  - scalar multiplying vectors by a positive number λ corresponds to increasing/decreasing the amplitude of the corresponding vibration by a factor of λ
  - scalar multiplication by a negative number λ corresponds to changing the amplitude of the vibration by a factor λ and reversing the *phase* of the vibration (this is the way noise cancelling headphones work)

Linear Combinations

The operations of vector and addition are readily composed with one another.
For example, if $\lambda, \nu \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then we can scalar multiply $\mathbf{a}$ by λ, and $\mathbf{b}$ by ν, and then add the two resulting vectors. We can then indicate the resulting vector as

$$\lambda \mathbf{a} + \nu \mathbf{b}$$

More generally,

Definition

A **linear combination** of n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a vector of the form

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n$$

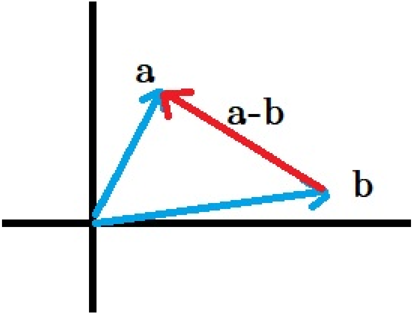
for some choice of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Vector Subtraction

A particularly useful linear combination of two vectors **a** and **b** is their vector difference

$$\mathbf{a} - \mathbf{b} \equiv \mathbf{a} + (-1)\mathbf{b}$$

Vector subtraction is used to indicate the *displacement* between two vectors. When the vectors $\mathbf{a}, \mathbf{b}$ are represented geometrically as directed line segments, $\mathbf{a} - \mathbf{b}$ corresponds to the directed line segment that goes from the tip of **b** to the tip of **a**.



The Zero Vector

Definition

The **zero vector** in $\mathbb{R}^n$ is the n-dimensional vector whose components are all 0's.

Thus, the zero vector in $\mathbb{R}^3$ is the vector

$$\mathbf{0} \equiv [0, 0, 0]$$

Just like with numbers, two vectors are the same **if and only if** their difference is zero:

$$\mathbf{a} = \mathbf{b} \qquad \Longleftrightarrow \qquad \mathbf{a} - \mathbf{b} = \mathbf{0}$$

Caution

Notationally, linear algebra looks a lot like regular algebra.
Such notation is useful because it allows us to exploit their similarity (e.g., both variables and vectors can be added).
But the objects themselves (variables vs. vectors) are very different animals.
And this can be confusing since we still use ordinary algebra within linear algebra.
This is why we try to use different fonts for vectors vs variables - but in hand written notes, it's not so easy to make the distinction between variables and vectors.
Some people, especially physicists, put little arrows above a letters to indicate vectors.