Math 3013 Lecture 3

January 14, 2022

Agenda

- Recap of Lectures 1 and 2
- The Dot Product
- The Basic Geometric Objects of Linear Algebra: points, lines, planes and hyperplanes

Recap of Lectures 1 and 2

- In this course, **vectors** are usually ordered lists of numbers
- $\mathbf{v} = [v_1, \ldots, v_n]$ is the usual way we'll denote a generic n-dimensional vector (each component v_i, $i = 1, \ldots, n$, is a real number).
- Occasionally, we depict vectors as directed line segments for visualization and intuition
- $\mathbb{R} \equiv$ the set of real numbers
- $\mathbb{R}^n =$ the set of n-dimensional (real) vectors

$$\mathbb{R}^n \equiv \{[v_1, v_2, \ldots, v_n] \mid v_1, v_2, \ldots, v_n \in \mathbb{R}\}$$

Two Fundamental Vector Operations

- **Vector Addition**

$$[v_1, v_2, \ldots, v_n] + [u_1, u_2, \ldots, u_n] = [v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n]$$

- **Scalar Multiplication**

$$\lambda \cdot [v_1, v_2, \ldots, v_n] = [\lambda v_1, \lambda v_2, \ldots, \lambda v_n]$$

Formal Properties of Fundamental Vector Operations of $\mathbb{R}^n$

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity of Vector Addition)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity of Vector Addition)
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ (Additive Identity) (Here $\mathbf{0}$ is the n-dimensional **zero vector** $[0, 0, \ldots, 0]$.)
4. $\mathbf{u}+(-1)\mathbf{u} = \mathbf{0}$ (Additive Inverses)
5. $c\,(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (Distributivity of Scalar Multiplication over Vector Addition)
6. $(c + d)\,\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (Distribution of Scalar Addition for Scalar Multiplication)
7. $c\,(d\mathbf{u}) = (cd)\,\mathbf{u}$ (Compatibility of Scalar Multiplication with Ordinary Multiplication of Numbers)
8. $1 \cdot \mathbf{u} = \mathbf{u}$ (Preservation of Scale)

Formal Properties, Cont'd

Each of these 8 identities is simple to prove.
E.g., to show

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \tag{1}$$

we simply calculate both sides using ordered lists of numbers for $\mathbf{u}$ and $\mathbf{v}$.
Explicitly, if $\mathbf{u} = [u_1, \ldots, u_n]$ and $\mathbf{u} = [v_1, \ldots, v_n]$, then

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= [u_1, \ldots, u_n] + [v_1, \ldots, v_n] \\
&= [u_1 + v_1, \ldots, u_n + v_n] \quad \text{(by def. of vector addition)} \\
&= [v_1 + u_1, \ldots, v_n + u_n] \quad \text{(commutativity of addition in } \mathbb{R}\text{)} \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}$$

□

Formal Properties, Cont'd

Later in the course, we'll see that **any set** V with two operations

$$\begin{aligned} + \quad &: \quad V \times V \to \quad \text{(Vector Addition)} \\ * \quad &: \quad \mathbb{R} \times V \to V \quad \text{(Scalar Multiplication)} \end{aligned}$$

obeying the same 8 identities is going to behave exactly like $\mathbb{R}^n$ as far as linear algebra goes.

Moreover, it will turn out that we will be able to calculate in V by exploiting this tight connection between V and $\mathbb{R}^n$.

The Dot Product : $\bullet : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

The vector dot product is an operation that ascribes a real number to a pair of vectors:
The dot product is used to extract geometric information from vectors

- Numerical/Linear Algebraic Version

$$\begin{aligned} [a_1, a_2, \ldots, a_n] \cdot [b_1, b_2, \ldots, b_n] &\equiv a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &\equiv \sum_{i=1}^{n} a_i b_i \end{aligned}$$

- Geometric Version:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta_{ab})$$

where

$$\begin{aligned} \|\mathbf{a}\| &= \text{length of the vector } \mathbf{a} \\ \theta_{ab} &= \text{angle between the vectors } \mathbf{a} \text{ and } \mathbf{b} \end{aligned}$$

Computing lengths and angles in n-dimensions

From the geometric rule

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta_{ab})$$

we can readily derive formulas for lengths and angles using the dot product:

$$\begin{aligned}
\|\mathbf{a}\| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \\
&= \sqrt{a_1^2 + \cdots a_n^2} \\
\theta_{ab} &= \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}}\sqrt{\mathbf{b} \cdot \mathbf{b}}}\right) \\
&= \cos^{-1}\left(\frac{a_1 b_1 + \cdots a_n b_n}{\sqrt{a_1^2 + \cdots a_n^2}\sqrt{b_1^2 + \cdots + b_n^2}}\right)
\end{aligned}$$

Example 1

Compute the length of $\mathbf{a} = [1, 2, -1, 1] \in \mathbb{R}^4$.

On the one hand, we have

$$\mathbf{a} \cdot \mathbf{a} = (1)\,(1) + (2)\,(2) + (-1)(-1) + (1)(1) = 6$$

while according to the geometric rule

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \cos(0) = \|\mathbf{a}\|^2$$

so we must have

$$\|\mathbf{a}\|^2 = 6 \quad \Rightarrow \quad \|\mathbf{a}\| = \sqrt{6}$$

Example 2

Compute the angle between $\mathbf{a} = [1, -1, 1, 1]$ and $\mathbf{b} = [2, 1, 1, -1]$

We have

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{(1)^2 + (-1)^2 + (1)^2 + (1)^2} = \sqrt{4} = 2 \\ \|\mathbf{b}\| &= \sqrt{\mathbf{b}\cdot\mathbf{b}} = \sqrt{(2)^2 + (1)^2 + (1)^2 + (-1)^2} = \sqrt{7} \end{aligned}$$

and

$$\mathbf{a}\cdot\mathbf{b} = (1)(2) + (-1)(1) + (1)(1) + (1)(-1) = 1$$

and so

$$1 = \mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos(\theta_{ab}) = 2\sqrt{7}\cos(\theta_{ab})$$

Solving for θ_{ab} yields

$$\theta_{ab} = \cos^{-1}\left(\frac{1}{2\sqrt{7}}\right) = 79.1^\circ$$

Vectors and Simple Geometric Objects

- Points $\longleftrightarrow$ single vectors

  Just as points in 3-dimensional space correspond to their coordinates $[x, y, z]$, points in an n-dimensional space correspond to elements of $\mathbb{R}^n$.

- Lines $\longleftrightarrow$ sets of vectors of the form

$$\ell = \{\mathbf{p}_0 + t\mathbf{d} \mid t \in \mathbb{R}\}$$

  (the line through $\mathbf{p}_0$ in the direction of $\mathbf{d}$)
  This is the principal way we will precribe lines in linear algebra.
  Other ways of prescribing lines

  - a line passing through the points $\mathbf{a}$ and $\mathbf{b}$

$$\ell = \{\mathbf{a} + t(\mathbf{b} - \mathbf{a}) \mid t \in \mathbb{R}\}$$

  - a line as a parameterized linear curve $\gamma : \mathbb{R} \to \mathbb{R}^n$

$$\ell = \{\gamma(t) = [a_1 + td_1, a_2 + td_2, \ldots, a_n + td_n] \mid t \in \mathbb{R}\}$$

    (note that components of $\gamma(t)$ are linear functions of the parameter t).

Higher Dimensional Objects

- Planes $\longleftrightarrow$ sets of vectors of the form

$$P = \{\mathbf{p}_0 + s\mathbf{d}_1 + t\mathbf{d}_2 \mid s, t \in \mathbb{R}\}$$

- Hyperplanes

$$H = \{\{\mathbf{p}_0 + t_1\mathbf{d}_1 + t_2\mathbf{d}_2 + \cdots + t_k\mathbf{d}_k \mid t_1, t_{2,} \ldots, t_k \in \mathbb{R}\}\}$$

Note that all examples considered previously are also hyperplanes (of lower dimension)

- Points : sets of the form $\{\mathbf{p}_0\}$
- Lines : sets of the form $\{\mathbf{p}_0 + t_1\mathbf{d}_1 \mid t_1 \in \mathbb{R}\}$
- Planes : sets of the form $\{\mathbf{p}_0 + t_1\mathbf{d}_1 + t_2\mathbf{d}_2 \mid t_1, t_2 \in \mathbb{R}\}$
- General Hyperplane : sets of the form $\{\{\mathbf{p}_0 + t_1\mathbf{d}_1 + t_2\mathbf{d}_2 + \cdots + t_k\mathbf{d}_k \mid t_1, t_{2,} \ldots, t_k \in \mathbb{R}\}\}$

Linear Equations and Hyperplanes

Definition

A **linear equation** in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{2}$$

The symbols $a_1, \ldots, a_n$ represent particular real numbers and are referred to as the **coefficients** of the **variables** $x_1, \ldots, x_n$. The symbol b on the right is also to be a real number and it is referred to as the **inhomogeneous term** in the equation.

Theorem

The solutions of a linear equation is always a hyperplane in $\mathbb{R}^n$.

Proof

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Using high school algebra we can readily solve an equation of the form (2) for x_n

$$x_n = \frac{b}{a_n} - \frac{a_1}{a_n}x_1 - \ldots - \frac{a_{n-1}}{a_n}x_{n-1}$$

and so a solution vector would be of the form

$$\begin{aligned}
\mathbf{x} &= \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ \frac{b}{a_n} - \frac{a_1}{a_n}x_1 - \ldots - \frac{a_{n-1}}{a_n}x_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{b}{a_n} \end{bmatrix} + \begin{bmatrix} x_1 \\ \vdots \\ 0 \\ -\frac{a_1}{a_n}x_1 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ x_{n-1} \\ -\frac{a_{n-1}}{a_n}x_{n-1} \end{bmatrix}
\end{aligned}$$

Proof, Cont'd

or

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{b}{a_n} \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ \vdots \\ 0 \\ -\frac{a_1}{a_n} \end{bmatrix} + \cdots + x_{n-1} \begin{bmatrix} 0 \\ \vdots \\ \\ -\frac{a_{n-1}}{a_n} \end{bmatrix} \\ &= \mathbf{p}_0 + x_1\mathbf{d}_1 + \cdots x_{n-1}\mathbf{d}_{n-1} \end{aligned}$$

WebAssign Problems

WebAssign Problem Set 1 must be submitted by 11:59 pm, Friday, January 21.