Lecture 7 : Solving Linear Systems

Math 3013
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Lecture 7:

Agenda:

1. Review: Linear Systems and their Solution Sets
2. Elementary Equation Operations that Preserve Solution Sets
3. Linear Systems and Augmented Matrices
4. Elementary Row Operations
5. Row Echelon Form and Reduced Row Echelon Form
6. The Row Reduction Algorithm

Solution Sets of $n \times m$ Linear Systems

Recall that a **hyperplane** is a set of vectors for the form

$$\{\mathbf{p}_0 + t_1\mathbf{v}_1 + \cdots + t_{m-1}\mathbf{v}_{m-1} \mid t_1, \ldots, t_{m-1} \in \mathbb{R}\}$$

and that

- points $\longleftrightarrow$ hyperplanes of the form $\{\mathbf{p}_0\}$
- lines $\longleftrightarrow$ hyperplanes of the form $\{\mathbf{p}_0 + t_1\mathbf{v}_1 \mid t_1 \in \mathbb{R}\}$
- planes $\longleftrightarrow$ hyperplanes of the form $\{\mathbf{p}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid t_1, t_2 \in \mathbb{R}\}$
- etc.

Theorem

The solution set of and $n \times m$ linear system (i.e. a system of n linear equations in m unknowns) is either

- *the empty set $\{\}$ (meaing there no solution at all) , or*
- *A hyperplane in $\mathbb{R}^m$ of dimension $\geq m - n$ (The strict equality holding so long as there are no redundacies amongst the equations)*

(We'll end up proving this theorem by our method of solution).

Solving Linear Systems

In high school, one learns to solve systems of equations by using the equations, one-by-one, to systematically reduce the number of variables to a minimal set (the free variables of the solution) upon which the other variables depend.

The essential idea of method of solution to be developed here will be to manipulate the equations (rather than the variables) until one obtains the equations of the solution.

Since we'll be manipulating equations, the first thing to explain is how one can modify sets of equations without changing their solutions.

Elementary Operations

Elementary Operations are operations we can perform on sets of equations that do not change their solution.
Consider the following 2 linear equations

$$x + y = 2 \tag{Eq1}$$

$$x - y = 1 \tag{Eq2}$$

Then

(i) we can change the order of equations

$$\left\{ \begin{array}{c} x + y = 2 \\ 2x - 2y = 4 \end{array} \right\} \xrightarrow{Eq_1 \longleftrightarrow Eq_2} \left\{ \begin{array}{c} 2x - 2y = 4 \\ x + y = 2 \end{array} \right\}$$

Elementary Operations, Cont'd

(ii) we can replace an equation by a non-zero scalar multiple of itself

$$\left\{ \begin{array}{c} x + y = 2 \\ 2x - 2y = 4 \end{array} \right\} \xrightarrow{Eq_2 \to \frac{1}{2} * Eq_2} \left\{ \begin{array}{c} x + y = 2 \\ x - y = 2 \end{array} \right\}$$

(iii) we can replace an equation by its sum with a multiple of another equation

$$\left\{ \begin{array}{c} x + y = 2 \\ 2x - 2y = 4 \end{array} \right\} \xrightarrow{Eq_2 \to Eq_2 + 2 * Eq_1} \left\{ \begin{array}{c} x + y = 2 \\ 4x = 6 \end{array} \right\}$$

Each set of equations on the right has exactly the same solutions as the original set.

Solving Linear Systems by Using Elementary Operations

I'll now demonstrate how one can solve a set equations by systematically converting the original set of equations to the equations of the solution **using only the following three operations:**

- $Eq_i \longleftrightarrow Eq_j$: Changing the order of the equations i and j
- $Eq_i \longrightarrow \lambda Eq_i$: Replacing the i^{th} equation by its multiple by a non-zero number λ
- $Eq_i \longrightarrow Eq_i + \lambda Eq_j$: Replacing an equation by its sum with a multiple of another equation

Example

Consider the following 2×2 linear system

$$\begin{aligned} x + y &= 1 \\ x - y &= 3 \end{aligned}$$

We have

$$\left\{ \begin{array}{c} x + y = 1 \\ x - y = 3 \end{array} \right\} \xrightarrow{Eq_2 \to Eq_2 + Eq_1} \left\{ \begin{array}{c} x + y = 1 \\ 2x = 4 \end{array} \right\}$$

$$\xrightarrow{Eq_2 \to \frac{1}{2} Eq_2} \left\{ \begin{array}{c} x + y = 1 \\ x = 2 \end{array} \right\}$$

$$\xrightarrow{Eq_1 \to Eq_1 - Eq_2} \left\{ \begin{array}{c} y = -1 \\ x = 2 \end{array} \right\}$$

$$\xrightarrow{Eq_1 \longleftrightarrow Eq_2} \left\{ \begin{array}{c} x = 2 \\ y = -1 \end{array} \right\} \quad \text{(the solution)}$$

The Matrix Method for Solving Linear Systems

Let's now reintroduce matrices into the game.
Recall that the data that goes into specifying an $n \times m$ linear system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1m}x_m &= b_1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

is an $n \times m$ matrix $\mathbf{A}$, an $m \times 1$ matrix of variables $\mathbf{x}$, and a $n \times 1$ matrix of numbers $\mathbf{b}$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \ldots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \quad , \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

In fact, the matrix equation $\mathbf{Ax} = \mathbf{b}$ is equivalent to the original linear system.

Augmented Matrices

Definition

The **augmented matrix** of an $n \times m$ linear system $\mathbf{Ax} = \mathbf{b}$ is the $n \times (m+1)$ matrix $[\mathbf{A} \mid \mathbf{b}]$ formed by adjoining the column vector $\mathbf{b}$ to the $n \times m$ matrix $\mathbf{A}$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right]$$

Augmented Matrices and Equations: Example

The augmented matrix of the 3×3 linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 1 \\ x_2 - 2x_3 &= 3 \end{aligned}$$

is

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

The Idea To Be Pursued

Schematically,

Step 1. linear system $\rightarrow$ augmented matrix $[\mathbf{A}|\mathbf{b}]$

Step 2. $[\mathbf{A} \mid \mathbf{b}] \rightarrow \left[\mathbf{A}' \mid \mathbf{b}'\right]$ = the augmented matrix of the solution

Step 3. $\left[\mathbf{A}'|\mathbf{b}'\right] \rightarrow$ the equations of solution

Step 4. Write down the solution set as a hyperplane

In the second step we will be using operations on augmented matrices that correspond to operations on equations that don't change their solution.

Elementary Row Operations

Since Elementary Operations change a set of equations they will also change the corresponding Augmented Matrix. In fact, the corresponding operations on augmented matrices can be implemented quickly and easily.

(i) Interchanging Equations $\longleftrightarrow$ Interchanging the corresponding rows of $[\mathbf{A}|\mathbf{b}]$

(ii) Multiplying an equation by a number $\lambda \neq 0 \quad \longleftrightarrow$ Scalar multiplying the corresponding row of $[\mathbf{A}|\mathbf{b}]$ by λ

(iii) Replacing and equation by its sum with a multiple of another equation $\longleftrightarrow$ replacing a row of $[\mathbf{A}|\mathbf{b}]$ with its vector sum with a scalar multiple of another row

Shorthand Notation for Elementary Row Operations

(i) $R_i \leftrightarrow R_j$
(row interchange)

(ii) $R_i \longrightarrow \lambda R_i$
(scalar multiplying a row by $\lambda \neq 0$)

(iii) $R_i \longrightarrow R_i + \lambda R_j$
(replacing i^{th} row by its sum with a multiple of the j^{th} row)

Using Elementary Row Operations to Solve a Linear System

$$\left\{\begin{array}{ll} x+y &= 3 \\ x-y &= 1 \end{array}\right\} \xrightarrow{\text{to augmented matrix}} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{back to equations}} \left\{\begin{array}{ll} x &= 2 \\ y &= 1 \end{array}\right\}$$

What's left to explain:

- The order and choice of operations used to reduce the augmented matrix of the original equations to the augmented matrix of the solution
- How to identify an augmented matrix as the augmented matrix of the solution

We'll answer the second question first.
(Once we know our destination, it'll be easier to explain how to get there.)

Digression: Pivots and Row Echelon Form

Definition

A **pivot** in the row of a matrix is the first non-zero entry of the row as one reads from left to right.

Example: In the following matrix the pivots (and only the pivots) have been underlined.

$$\begin{bmatrix} 0 & \underline{2} & 1 & 0 & 1 \\ \underline{1} & 0 & 2 & 1 & 1 \\ 0 & 0 & \underline{3} & 1 & -1 \end{bmatrix}$$

Definition

A matrix is in **row echelon form** if

- the pivots in upper rows always occur to the left of pivots in lower rows
- any row that contains only zeros appears at the bottom of the matrix

Row Echelon Form : Examples

$$\begin{bmatrix} \underline{1} & 0 & 2 & 1 \\ 0 & \underline{3} & 1 & 1 \\ 0 & 0 & 0 & \underline{2} \end{bmatrix} \qquad \text{in R.E.F.}$$

$$\begin{bmatrix} 0 & \underline{2} & 1 & 0 & 1 \\ \underline{1} & 0 & 2 & 1 & 1 \\ 0 & 0 & \underline{3} & 1 & -1 \end{bmatrix} \qquad \text{not in R.E.F.}$$

N.B. In the second matrix the pivot in the first row is to the right of the pivot in the second row)

Reduced Row Echelon Form

Definition

A matrix is in **reduced row echelon form** if

- it is in row echelon form
- the pivots are always equal to 1
- if a column contains a pivot, then the pivot is the only non-zero entry in that column

Examples of Matrices in Reduced Row Echelon Form

Example:

$$\begin{bmatrix} \underline{1} & 0 & 2 & 1 \\ 0 & \underline{1} & 1 & 1 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$$

is **almost** in Reduced Row Echelon Form. It satisfies the first two conditions (it is in R.E.F. and all pivots equal 1). However, the fourth column contains non-zero entries besides the pivot in the third row.

Example:

$$\begin{bmatrix} \underline{1} & 0 & 2 & 0 \\ 0 & \underline{1} & 1 & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$$

is in Reduced Row Echelon Form.

Augmented Matrices in Reduced Row Echelon Form

We are interested in converting augmented matrices to Reduced Row Echelon Form.
Because an augmented matrix in R.R.E.F. is interpretable as the augmented matrix corresponding to the solution of a system of linear equations.
Example: The following matrix is in R.R.E.F.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right] \longleftrightarrow \left\{\begin{array}{ccc} x_1 & = & 2 \\ x_2 & = & 1 \\ x_3 & = & 4 \end{array}\right\}$$

Examples of Row Echelon Form and Reduced Row Echelon Form

$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	R.E.F. but not R.R.E.F
$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix}$	not R.E.F
$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 \end{bmatrix}$	R.E.F. but not R.R.E.F.

Examples of R.E.F. and R.R.E.F., Cont'd

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ R.R.E.F

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ R.R.E.F.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ R.R.E.F.

Solving Linear Systems via Row Reduction

A basic outlike of our method is as follows

- Convert the equations to an augmented matrix $[\mathbf{A} \mid \mathbf{b}]$.
- Use elementary row operations to convert $[\mathbf{A} \mid \mathbf{b}]$ to its Reduced Row Echelon Form $[\mathbf{A}' \mid \mathbf{b}']$.
- The equations of the solution are obtained by converting the RREF matrix $[\mathbf{A}' \mid \mathbf{b}']$ back into equations.
- Reinterprete the solution equations as prescribing the solutions as vectors.