Lecture 8 : Row Reduction

Math 3013
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Lecture 8:

Agenda:

1. Recap of Lecture 7
   - Augmented Matrices
   - Program for Solving Linear Systems
   - Row Echelon Form and Reduced Row Echelon Form
   - Elementary Row Operations
2. The Row Reduction Algorithm

Augmented Matrices

Definition

The **augmented matrix** of an $n \times m$ linear system $\mathbf{Ax} = \mathbf{b}$ is the $n \times (m+1)$ matrix $[\mathbf{A} \mid \mathbf{b}]$ formed by adjoining the column vector $\mathbf{b}$ to the $n \times m$ matrix $\mathbf{A}$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right]$$

Elementary Row Operations

Elementary Row Operations are modifications of an augmented matrix that leave the solution sets of the corresponding linear systems unchanged.
There are 3 basic row operations

(i) $R_i \leftrightarrow R_j$
(row interchange)

(ii) $R_i \longrightarrow \lambda R_i$
(scalar multiplying a row by $\lambda \neq 0$)

(iii) $R_i \longrightarrow R_i + \lambda R_j$
(replacing i^{th} row by its sum with a multiple of the j^{th} row)

The Method To Be Developed

Schematically,

Step 1. Write down the augmented matrix $[\mathbf{A}|\mathbf{b}]$ corresponding to the original linear system.

Step 2. Use the Elementary Row Operations to convert $[\mathbf{A}|\mathbf{b}]$ to its Reduced Row Echelon Form $\left[\mathbf{A}'|\mathbf{b}'\right]$ (the augmented matrix of corresponding to the solution)

Step 3. Convert $\left[\mathbf{A}'|\mathbf{b}'\right]$ back to equations to get the equations of solution

Step 4. Write down the solution set as a hyperplane

Today, we will be focusing on Step 2.

Nomenclature from Lecture 7

- The first non-zero entry of a matrix row is called the **pivot** of that row.
- A matrix is in **Row Echelon Form** if the pivots in lower rows always occur to the right of the pivots in upper rows.
- A matrix in **Reduced Row Echelon Form** if it is in Row Echelon Form and, in addition,
  - all pivots $=1$
  - directly above and below the pivots, only 0's appear.

An augmented matrix in R.R.E.F. will correspond to the augmented matrix of the solution of a linear system.
The process by which we convert an augmented matrix $[\mathbf{A}|\mathbf{b}]$ to its (unique) R.R.E.F. $\left[\mathbf{A}'|\mathbf{b}'\right]$ $\rightarrow$ is called **Row Reduction**.

Example: reducing a matrix to Row Echelon Form

$$\begin{bmatrix} 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\mathcal{R}_2 \longleftrightarrow \mathcal{R}_4} \begin{bmatrix} 0 & 1 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\mathcal{R}_1 \longleftrightarrow \mathcal{R}_2} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\mathcal{R}_3 \longleftrightarrow \mathcal{R}_3 - \mathcal{R}_1} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\mathcal{R}_3 \longleftrightarrow \mathcal{R}_3 + \mathcal{R}_2} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction - Stage 1: Converting a matrix to Row Echelon Form

- If there are any zero rows in the matrix use the row interchange operation $\mathcal{R}_i \longleftrightarrow \mathcal{R}_j$ to move them to the bottom of the matrix.
- If the pivot in the first row is not a left-most pivot in the matrix, interchange the first row with a lower row that does have a left-most pivot.
- If there are any non-zero entries in rows $\mathcal{R}_2, \ldots, \mathcal{R}_n$ that lie directly below the pivot in the first row, use the elementary operation $\mathcal{R}_i \rightarrow \mathcal{R}_i - \frac{a_{ij}}{a_{1j}}\mathcal{R}_1$ to clear out that non-zero entry (here j corresponds to the column index of the pivot in the first row).
- Repeat preceding step until all entries directly below the pivot in the first row have been zero-ed out.

Row Reduction - Stage 1: Converting a matrix to Row Echelon Form, Cont'd

- Repeat the process above, using the second row rather than the first row: Thus,
  - If there are (new) zero rows, move them to the bottom of the matrix using row interchanges.
  - If there is row below $\mathcal{R}_2$ that has a pivot to the left of that of $\mathcal{R}_2$, exchange that row with $\mathcal{R}_2$.
  - If there are non-zero entries below the pivot in $\mathcal{R}_2$, use the operation $\mathcal{R}_i \to \mathcal{R}_i - \frac{a_{ij}}{a_{2j}}\mathcal{R}_2$ to clear out the non-zero entries below the pivot in $\mathcal{R}_2$
  - etc.
- Repeat this process, moving down the matrix row-by-row.
- Stage 1 terminates when one reaches the bottom matrix, at which point the matrix will be in Row Echelon Form.

Row Reduction - Stage 2: converting pivots to 1's

If any row of the matrix has a pivot has a pivot with value $\lambda \neq 1$ then use the row rescaling operation $\mathcal{R}_i \to \frac{1}{\lambda}\mathcal{R}_i$ to convert the pivot value of that row to 1.

Example: Stage 2

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\substack{\mathcal{R}_1 \to \frac{1}{2}\mathcal{R}_1 \\ \mathcal{R}_2 \to \frac{1}{3}\mathcal{R}_2 \\ \mathcal{R}_3 \to -\mathcal{R}_3}} \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Row Reduction - Stage 3 : zero-out entries above pivots

This stage is similar to Stage 1, except instead of working with an upper row to zero out the entries below a pivot, one uses lower rows to eliminate entries above the pivots in the lower rows:

- If the last row $\mathcal{R}^n$ has a pivot in column j, and the i^{th} row, $i > n$ has a non-zero entry a_{ij} in its j^{th} slot, use the row operation $\mathcal{R}_i \to \mathcal{R}_i - a_{ij}\mathcal{R}_n$ to clear out that entry. (Here we're assuming that the pivot in the last row has already been rescaled to 1.)
- Repeat the first step until only zero's occur above the pivot in the last row.
- Move on to the second to last row and use the same process as above to eliminate the non-zero entries above the pivot in the second to last row.
- Continue moving up through the matrix row-by-row, until the entries above each pivot have all been zero-ed out.

At the end of the Stage 3, the matrix will be in Reduced Row Echelon Form.

Example: Stage 3 Row Reduction

$$\begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[]{\substack{\mathcal{R}_1 \to \mathcal{R}_1 - \frac{1}{2}\mathcal{R}_3 \\ \mathcal{R}_2 \to \mathcal{R}_3 - 2\mathcal{R}_3}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\mathcal{R}_1 \to \mathcal{R}_1 - 2\mathcal{R}_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Solving a Linear System via Row Reduction

Consider the following system of equations

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 2 \\ 2x_1 - x_2 - x_3 &= -3 \end{aligned}$$

Step 0: Form the augmented matrix for the system:

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & -1 & -3 \end{array}\right]$$

Step 1 of Row Reduction:

Starting from the upper left hand corner, clear out the entries below the pivots

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & -1 & -3 \end{array}\right] \xrightarrow[\mathcal{R}_3 \to \mathcal{R}_3 - 2\mathcal{R}_1]{\mathcal{R}_2 \to \mathcal{R}_2 - \mathcal{R}_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & -3 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & -3 & 1 & -3 \end{array}\right] \xrightarrow{\mathcal{R}_2 \to -\frac{1}{2}\mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & 1 & -3 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 + 3\mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -6 \end{array}\right]$$

Note that the last augmented matrix is in Row Echelon Form (R.E.F.).

Step 2 of Row Reduction:

Convert pivots to 1's

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -6 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to -\frac{1}{2}\mathcal{R}_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 3 of Row Reduction:

Starting from the lower right hand corner,

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[]{\substack{\mathcal{R}_1 \to \mathcal{R}_1 + \mathcal{R}_3 \\ \mathcal{R}_2 \to \mathcal{R}_2 + \mathcal{R}_3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\mathcal{R}_1 \to \mathcal{R}_1 - \mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Note that the last augmented matrix is in Reduced Row Echelon Form

Step 4:

Convert the augmented matrix in R.R.E.F. back to equations:

$$\left[\mathbf{A}'|\mathbf{b}'\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left.\begin{array}{rcl} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 & = & 1 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 & = & 2 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 & = & 3 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 2 \\ x_3 & = & 3 \end{array}\right.$$

Step 5:

Reformulate the solution as a vector or a set of vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In this example, we have a unique vector solution.

Summary: Solving Linear Systems using Augmented Matrices and Row Reduction

Step 1: Convert equations to an augmented matrix $[\mathbf{A} \mid \mathbf{b}]$
Step 2: Row reduce $[\mathbf{A} \mid \mathbf{b}]$ to Row Echelon Form $[\mathbf{A}' \mid \mathbf{b}']$
Step 3: Row reduce $[\mathbf{A}' \mid \mathbf{b}']$ to its Reduced Row Echelon Form $[\mathbf{A}'' \mid \mathbf{b}'']$
Step 4: Convert $[\mathbf{A}'' \mid \mathbf{b}'']$ back to equations
Step 5: Reformulate the solution as a vector or a set of vectors.

An Example With Hidden Redundancies

Consider the linear system Solve the following system of equations using Gaussian reduction.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 4 \end{aligned}$$

Naively,

$$\begin{aligned} \text{dim solution space} &= \# \textit{free variables} \\ &= \# \textit{variables} - \# \textit{equations} \\ &= 3 - 3 = 0 \end{aligned}$$

and so

$$0 \text{ free variables} \quad \Rightarrow \quad \text{unique solution}$$

However, as we shall see, in this example we actually have infinitely many solutions.

Step 1: Convert to Augmented Matrix

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 4 \end{aligned}$$

$$\Downarrow$$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 3 & 4 \end{array}\right]$$

Step 2: $[\mathbf{A} \mid \mathbf{b}] \to R.E.F.$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 3 & 4 \end{array}\right] \xrightarrow[\mathcal{R}_3 \to \mathcal{R}_3 - 3\mathcal{R}_1]{\mathcal{R}_2 \to \mathcal{R}_2 - \mathcal{R}_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (R.E.F.)$$

Step 3: $R.E.F. \to R.R.E.F.$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\mathcal{R}_2 \to -\frac{1}{2}\mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\mathcal{R}_1 \to \mathcal{R}_1 - \mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (R.R.E.F.)$$

Step 4: Back to equations:

$$R.R.E.F.([\mathbf{A}|\mathbf{b}]) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{rcl} 1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 & = & 1 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 & = & 1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 & = & 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{ccc} x_1 + x_3 & = & 1 \\ x_2 & = & 1 \\ 0 & = & 0 \end{array}\right.$$

Note that the last equation puts no restriction on x_1, x_2, x_3. So our original system is actually equivalent to a system of 2 equations in three unknowns.

Step 5: Reformulate the solution as a vector or a set of vectors

The "solution equations" are thus

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 &= 1 \end{aligned}$$

Something new happens here; the first equation in the "solution" involves two variables.
To deal with this situtation, we will now make a distinction between **free variables** and **fixed variables**

Definition

Let $[\mathbf{A}'' \mid \mathbf{b}'']$ be the **R.R.E.F.** of the augmented matrix of an $n \times m$ linear system.

- We will regard a variable x_i as a **fixed variable** if there is a pivot in the i^{th} column of $[\mathbf{A}'' \mid \mathbf{b}'']$
- We will interpret x_i as a **free variable** if the i^{th} column of $[\mathbf{A}'' \mid \mathbf{b}'']$ does contain a pivot.

Fixed Varibles vs. Free Variables

$$\begin{array}{rcl} \text{fixed variables} & \iff & \text{columns in RREF with pivots} \\ \text{free variables} & \iff & \text{columns in RREF without pivots} \end{array}$$

In the current example, we have

$$\left[\mathbf{A}'' \mid \mathbf{b}''\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so

$$\begin{array}{rcl} x_1, x_2 & \rightarrow & \text{fixed variables} \\ x_3 & \rightarrow & \text{free variable} \qquad \text{(since there is no pivot in the } 3^{rd} \text{ column} \end{array}$$

Solution Vectors

To write down the solutions as vectors, we first rewrite the equations corresponding to $[\mathbf{A}'' \mid \mathbf{b}'']$, moving the free variables to the right hand side

$$\left.\begin{array}{rcl} x_1 + x_3 & = & 1 \\ x_2 & = & 1 \\ 0 & = & 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{lcl} x_1 & = & 1 - x_3 \\ x_2 & = & 1 \\ 0 & = & 0 \end{array}\right.$$

Step 5, cont'd

We next write down a typical solution vector, using the equations above to express the fixed variables x_1, x_2 in terms of the free variable x_3

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 1 \\ x_3 \end{bmatrix}$$

Finally, we expand this vector solution in terms of the free parameter x_3 :

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad , \qquad x_3 \in \mathbb{R}$$

Note that this form implies that every solution vector lives on the line

$$\ell = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

and so, in particular, we have infinitely many solutions

Example: a linear system with an internal contradiction

Consider

$$\begin{aligned} x + y + z &= 2 \\ x + 2y - z &= 3 \\ y - 2z &= 4 \end{aligned}$$

Note that if subtract the first equation from the second equation we get

$$\begin{array}{rcr} x + 2y - z & = & 3 \\ -(x + y + z) & = & -2 \\ \hline y - 2z & = & 1 \end{array}$$

which contradicts the third equation.
Thus, we should expect to find **no solution** to this system

Let's go ahead and apply our row reduction method and see what happens in this case.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow{\mathcal{R}_2 \to \mathcal{R}_2 - \mathcal{R}_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Now recall: Solution Sets Are Unchanged by Row Reduction. The equation corresponding the last row of the last augmented matrix is

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 3 \quad \Rightarrow \quad 0 = 3 \qquad \text{a contradiction}$$

So the row reduction process has exposed the internal contradiction in the original set of equations.

Summary: Redundant or Contradictory Equations and Row Reduction

In general, during the course of row reduction of an augmented matrix $[\mathbf{A} \mid \mathbf{b}]$,

- if a zero rows appears $\implies$ there is redundancy in the original set of equations
- if a pivot appears in the last column $\implies$ there is a contradiction in the original set of solutions

Thus, **row reduction exposes hidden redundancies and hidden contradictions**

In particular, whenever you end up with a pivot in last column, you may as well stop row reducing because **the original linear system has no solution**.

Review: Solving Linear Equations via Row Reduction of Augmented Matrices

Step 1: Convert to Augmented Matrix

$$\left.\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1m}x_m & = & b_1 \\ a_{21}x_1 + \cdots + a_{2m}x_m & = & b_2 \\ & \vdots & \\ a_{n1}x_1 + \cdots + a_{nm}x_m & = & b_n \end{array}\right\}$$

$$\Longrightarrow [\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right]$$

Step 2: Row Reduce to Reduced Row Echelon Form

Elementary Row Operations : operations that don't change the solution set of the linear system

- $\mathcal{R}_i \leftrightarrow \mathcal{R}_j$ (interchange i^{th} and j^{th} row)
- $\mathcal{R}_i \to \lambda\mathcal{R}_i$ (scalar multipliy i^{th} row by $\lambda \in \mathbb{R}$
- $\mathcal{R}_i \to \mathcal{R}_i + \lambda\mathcal{R}_j$ (replace row i with its sum with a multiple of row j)

Row Echelon Form
$$\begin{bmatrix} 0 & \underline{*} & * & * & * & * \\ 0 & 0 & \underline{*} & * & * & * \\ 0 & 0 & 0 & 0 & \underline{*} & * \\ 0 & 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form
$$\begin{bmatrix} 0 & \underline{1} & 0 & * & 0 & 0 \\ 0 & 0 & \underline{1} & * & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Remarks on Row Reduction

- Row Reduction effectively eliminates redundancies amongst equations (hidden redundancies show up as zero rows during row reduction).
- Row Reduction exposes contradictions amongst equations (if pivot occurs in last column of $[\mathbf{A} \mid \mathbf{b}]$ during row reduction, then original set of equations have no solution)
- The number and position of pivots is fixed once you reach Row Echelon Form
- You can identify the number of free variables in the solution, i.e. the dimension of the solution set, at the REF stage of row reduction
- You can tell whether or not there is a solution at the REF stage of row reduction

Step 3: Convert Back to Equations Distinguishing Fixed Variables From Free Variables

Fixed Variables : variables corresponding to columns of the RREF of $[\mathbf{A} \mid \mathbf{b}]$ that **do have** a pivot entry
Free Variables : variables corresponding to columns of the RREF that **do not have** a pivot entry
Example:

$$\text{RREF} \qquad \left[\begin{array}{cccccc|c} 0 & \underline{1} & 0 & 2 & 0 & 3 & 3 \\ 0 & 0 & \underline{1} & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{rcl} x_2 + 2x_4 + 3x_6 & = & 3 \\ x_3 + x_4 + 2x_6 & = & 2 \\ x_5 - 2x_6 & = & 1 \\ 0 & = & 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} x_2 & = & 3 - 2x_4 - 3x_6 \\ x_3 & = & 2 - x_4 - 2x_6 \\ x_5 & = & 1 + 2x_6 \end{array}\right.$$

Step 4 : Write Down Solution as a Point, Line, Plane, or Hyperplane of Vectors

Formulate a solution vector using the equations for the fixed variables and leaving the free variable components alone and expand solution vector in terms of the free variables

$$\begin{aligned}
x_1 &= \textit{free} \\
x_2 &= 3 - 2x_4 - 3x_6 \\
x_3 &= 2 - x_4 - 2x_6 \\
x_4 &= \textit{free} \\
x_5 &= 1 + 2x_6 \\
x_6 &= \textit{free}
\end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 3 - 2x_4 - 3x_6 \\ 2 - x_4 - 2x_6 \\ x_4 \\ 1 + 2x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -3 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Example: A Linear System with a Redundancy

Consider

$$\begin{aligned} x_1 + x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 3 \\ x_1 + x_2 + 2x_4 &= 4 \end{aligned}$$

Note Eq3 = Eq1 + Eq2, so Eq3 is already implied by Eq1 and Eq1; it thus amounts to a redundant condition on x_1, x_2, x_3, x_4.

Example, Cont'd

Step 1: Convert to Augmented Matrix

$$\left.\begin{array}{rcl} x_1 + x_3 + x_4 & = & 1 \\ x_2 - x_3 + x_4 & = & 3 \\ x_1 + x_2 + 2x_4 & = & 4 \end{array}\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 1 & 1 & 0 & 2 & 4 \end{array}\right]$$

Example with Redundancy, Cont'd

Step 2: Row Reduce to R.R.E.F.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 1 & 1 & 0 & 2 & 4 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (\textit{REF} \text{ and } \textit{RREF})$$

Example with Redundancy, Cont'd

Step 3: Solution as Equations

Note: x_3 and x_4 will be free variables in the solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{\begin{array}{ccc} x_1 + x_3 + x_4 & = & 1 \\ x_2 - x_3 + x_4 & = & 3 \\ 0 & = & 0 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{ccc} x_1 & = & 1 - x_3 - x_4 \\ x_2 & = & 3 + x_3 - x_4 \end{array}\right\}$$

Example with Redundancy, Cont'd

Step 4: Expand Solution Vector w.r.t. the Free Variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_3 - x_4 \\ 3 + x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is thus a 2-dimensional plane inside $\mathbb{R}^4$.

A Linear System with a Contradiction

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ x_1 - x_2 - x_3 + x_4 &= 2 \\ 2x_1 + 2x_4 &= 2 \end{aligned}$$

Note that the sum of the first two equations is

$$2x_1 + 2x_4 = 4$$

which contradicts the last equation. So there cannot be a valid solution to this system.

Example with a Contradiction, Cont'd

Row Reduction:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 1 & 2 \\ 2 & 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow[\mathcal{R}_3 \to \mathcal{R}_3 - 2\mathcal{R}_1]{\mathcal{R}_2 \to \mathcal{R}_2 + \mathcal{R}_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Example with Contradiction: Row Echelon Form

$$R.E.F. \qquad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Last row says

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = -2$$

or

$$0 = -2 \qquad \text{(a contradiction)}$$

We conclude: **there is no solution to the original linear system**.