Lecture 9 : Solving Linear Systems via Row Reduction

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Lecture 9: Solving Linear Systems via Row Reduction

Agenda:

1. Recap: The Linear Algebraic Method for Solving Linear Systems
2. From Augmented Matrices in Reduced Row Echelon Form to Solution Hyperplanes
3. Examples

Linear Systems, Augmented Matrices and Row Reduction

The Row Reduction Method:

Step 1. Write down the augmented matrix $[\mathbf{A}|\mathbf{b}]$ corresponding to the original linear system.

Step 2. Use the Elementary Row Operations to convert $[\mathbf{A}|\mathbf{b}]$ to its Reduced Row Echelon Form $\left[\mathbf{A}'|\mathbf{b}'\right]$ (the augmented matrix of corresponding to the solution)

Step 3. Convert $\left[\mathbf{A}'|\mathbf{b}'\right]$ back to equations to get the equations of solution

Step 4. Write down the solution set as a hyperplane

We've pretty much covered all but the last step of this method. Today, I'll show you how to carry out the last step and give you plenty of examples of the whole process.

Recall the connection between linear systems and their augmented matrices

$$\left\{ \begin{array}{ccc} a_{11}x_1 + \cdots + a_{1m}x_m & = & b_1 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m & = & b_n \end{array} \right\} \iff \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array} \right]$$

Recall also that by using **elemementary row operations** the augmented matrix of a linear system can be systematically converted into a unique augmented matrix in **Reduced Row Echelon Form** and that this augmented matrix in R.R.E.F. is interpetable as the augmented matrix of the solution.
I'll now show you how to use the augmented matrix in R.R.E.F. to write down the corresponding solution hyperplane.

Fixed Variables and Free Variables

Whenever one has fewer (consistent) equations than unknowns, there will always be infinitely many solutions. So is more apt to speak of the number of **free variables** in the solution (which will be the dimension of the solutions set). For example, consider the equation

$$x + y = 5$$

This equation can be rewritten as

$$y = 5 - x$$

In the latter form, we interpret the value of the variable y as being fixed by the value of the variable x. In this situation, we say that x is a **free variable** in the solution and that y is a **fixed variable** in the solution (since it is fixed by the value of x).

Of course, we could have also written the equation as

$$x = 5 - y$$

and in this case, we would regard y as the free variable in the solution and x as the fixed variable.

Fixed Variables and Free Variables via Augmented Matrices

What nice about the Row Reduction method of solving linear systems is that the augmented matrix in R.R.E.F. naturally informs us which variables should be consided as fixed variables and which variables should be considered as free variables.

Let $[\mathbf{A}|\mathbf{b}]$ be an augmented matrix in R.R.E.F. Since it is, in particular, in R.E.F., each column of $[\mathbf{A}|\mathbf{b}]$ has at most one pivot. We say that a variable x_i is a

- **fixed variable** if the i^{th} column of $[\mathbf{A}|\mathbf{b}]$ contains a pivot; or that
- **free variable** if the i^{th} column of $[\mathbf{A}|\mathbf{b}]$ does not contain a pivot.

Example

Consider the following augmented matrix in R.R.E.F.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

From the discussion above:

pivots in columns 1,3 and 5 $\Rightarrow$ x_1, x_3, x_5 are fixed variables

no pivots in columns 2 and 4 $\Rightarrow$ x_2 and x_4 free variables

Example, Cont'd

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Indeed when we convert the augmented matrix above back to equations

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 1 \\ x_3 + x_4 &= 2 \\ x_5 &= 3 \end{aligned}$$

and then move the "free variables" to the right hand side, we get

$$\begin{aligned} x_1 &= 1 - 2x_2 - x_4 \\ x_4 &= 2 - x_4 \\ x_5 &= 3 \end{aligned}$$

We thus end up with equations expressing each of the fixed variables in terms of the free variables and constants.

Example, Cont'd

Suppose now we try to write down a solution vector, substituting where we can, for the fixed variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 - 2x_2 - x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \\ 3 \end{bmatrix}$$

This shows that the solution vectors only depend the free variables x_2 and x_4.

Example, Cont'd

Moreover, having expressed solution vectors in terms of the free variables, it is now easy to construct the solution hyperplane.
To achieve this, all we have to do is expand the solution vector in terms of the free variables:

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} 1 - 2x_2 - x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 0 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Example, Cont'd

This last equation

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

exhibits $\mathbf{x}$ a vector living on the hyperplane

$$\mathcal{H} = \{\mathbf{p}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid t_1, t_2 \in \mathbb{R}\}$$

where

$$\mathbf{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \quad , \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Summary: Going for Augmented Matrix $[\mathbf{A}|\mathbf{b}]$ in R.R.E.F. to Solution Hyperplane

- Identify variables corresponding to columns with pivots as fixed variables and variables corresponding to columns without pivots as free variables.
- Write down the equations corresponding to $[\mathbf{A}|\mathbf{b}]$
- Move the free variables to the right hand sides of the equations to get equations expressing each of the fixed variables in terms of the free variables.
- Write down a generic solution vector, replacing the fixed variables by their expressions in terms of the free variables.
- Expand the solution vector in terms of the free parameters to identify the initial point $\mathbf{p}_0$ and the spanning directions $\mathbf{v}_1, \mathbf{v}_2, \ldots$ of the hyperplane expression for the solution.

An Example With Hidden Redundancies

Consider the following linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 4 \end{aligned}$$

Naively,

$$\begin{aligned} \text{dim solution space} &= \#\textit{variables} - \#\textit{equations} \\ &= 3 - 3 = 0 \end{aligned}$$

and so one would think

$$0 \text{ free variables} \quad \Rightarrow \quad \text{unique solution}$$

However, as we shall see, in this example we actually have infinitely many solutions.

Step 1: Convert to Augmented Matrix

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 4 \end{aligned}$$

$$\Downarrow$$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 3 & 4 \end{array}\right]$$

Step 2: $[\mathbf{A} \mid \mathbf{b}] \to R.E.F.$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 3 & 4 \end{array}\right] \xrightarrow[\mathcal{R}_3 \to \mathcal{R}_3 - 3\mathcal{R}_1]{\mathcal{R}_2 \to \mathcal{R}_2 - \mathcal{R}_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (R.E.F.)$$

Step 3: $R.E.F. \to R.R.E.F.$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\mathcal{R}_2 \to -\frac{1}{2}\mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\mathcal{R}_1 \to \mathcal{R}_1 - \mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (R.R.E.F.)$$

Step 4: Back to equations:

$$R.R.E.F.([\mathbf{A}|\mathbf{b}]) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{rcl} 1\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 & = & 1 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 & = & 1 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 & = & 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{ccc} x_1 + x_3 & = & 1 \\ x_2 & = & 1 \\ 0 & = & 0 \end{array}\right.$$

Note that the last equation puts no restriction on x_1, x_2, x_3. So our original system is actually equivalent to a system of 2 equations in three unknowns.

Step 5: Reformulate the solution set as a hyperplane

Recall

fixed variables	$\Longleftrightarrow$	columns in RREF with pivots
free variables	$\Longleftrightarrow$	columns in RREF without pivots

In the current example, we have

$$\left[\mathbf{A}'' \mid \mathbf{b}''\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so

$x_1, x_2 \quad \rightarrow \quad$ fixed variables

$x_3 \quad \rightarrow \quad$ free variable (since there is no pivot in the 3^{rd} column

Solution Vectors

To write down the solutions as vectors, we first rewrite the equations corresponding to $[\mathbf{A}'' \mid \mathbf{b}'']$, moving the free variables to the right hand side

$$\left.\begin{array}{rcl} x_1 + x_3 & = & 1 \\ x_2 & = & 1 \\ 0 & = & 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{lcl} x_1 & = & 1 - x_3 \\ x_2 & = & 1 \\ 0 & = & 0 \end{array}\right.$$

Step 5, cont'd

We next write down a typical solution vector, using the equations above to express the fixed variables x_1, x_2 in terms of the free variable x_3

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 1 \\ x_3 \end{bmatrix}$$

Finally, we expand this vector solution in terms of the free parameter x_3 :

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad x_3 \in \mathbb{R}$$

Note that this form implies that every solution vector lives on the line

$$\ell = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

and so, in particular, we have infinitely many solutions

Example: a linear system with an internal contradiction

Consider

$$\begin{aligned} x + y + z &= 2 \\ x + 2y - z &= 3 \\ y - 2z &= 4 \end{aligned}$$

Note that if subtract the first equation from the second equation we get

$$\begin{array}{rcr} x + 2y - z & = & 3 \\ -(x + y + z) & = & -2 \\ \hline y - 2z & = & 1 \end{array}$$

which contradicts the third equation.
Thus, we should expect to find **no solution** to this system

Let's go ahead and apply our row reduction method and see what happens in this case.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow{\mathcal{R}_2 \to \mathcal{R}_2 - \mathcal{R}_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Now recall: Solution Sets Are Unchanged by Row Reduction. The equation corresponding the last row of the last augmented matrix is

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 3 \quad \Rightarrow \quad 0 = 3 \qquad \text{a contradiction}$$

So the row reduction process has exposed the internal contradiction in the original set of equations.

Summary: Redundant or Contradictory Equations and Row Reduction

In general, during the course of row reduction of an augmented matrix $[\mathbf{A} \mid \mathbf{b}]$,

- if a zero rows appears $\Longrightarrow$ there is redundancy in the original set of equations
- if a pivot appears in the last column $\Longrightarrow$ there is a contradiction in the original set of solutions

Thus, **row reduction exposes hidden redundancies and hidden contradictions**

In particular, whenever you end up with a pivot in last column, you may as well stop row reducing because **the original linear system has no solution**.

Review: Solving Linear Equations via Row Reduction of Augmented Matrices

Step 1: Convert to Augmented Matrix

$$\left.\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1m}x_m & = & b_1 \\ a_{21}x_1 + \cdots + a_{2m}x_m & = & b_2 \\ & \vdots & \\ a_{n1}x_1 + \cdots + a_{nm}x_m & = & b_n \end{array}\right\}$$

$$\Longrightarrow [\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right]$$

Step 2: Row Reduce to Reduced Row Echelon Form

Elementary Row Operations : operations that don't change the solution set of the linear system

- $\mathcal{R}_i \leftrightarrow \mathcal{R}_j$ (interchange i^{th} and j^{th} row)
- $\mathcal{R}_i \to \lambda\mathcal{R}_i$ (scalar multipliy i^{th} row by $\lambda \in \mathbb{R}$
- $\mathcal{R}_i \to \mathcal{R}_i + \lambda\mathcal{R}_j$ (replace row i with its sum with a multiple of row j)

Row Echelon Form
$$\begin{bmatrix} 0 & \underline{*} & * & * & * & * \\ 0 & 0 & \underline{*} & * & * & * \\ 0 & 0 & 0 & 0 & \underline{*} & * \\ 0 & 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form
$$\begin{bmatrix} 0 & \underline{1} & 0 & * & 0 & 0 \\ 0 & 0 & \underline{1} & * & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3: Convert Back to Equations Distinguishing Fixed Variables From Free Variables

Fixed Variables : variables corresponding to columns of the RREF of $[\mathbf{A} \mid \mathbf{b}]$ that **do have** a pivot entry
Free Variables : variables corresponding to columns of the RREF that **do not have** a pivot entry
Example:

$$\text{RREF} \quad \left[\begin{array}{cccccc|c} 0 & \underline{1} & 0 & 2 & 0 & 3 & 3 \\ 0 & 0 & \underline{1} & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{rcl} x_2 + 2x_4 + 3x_6 & = & 3 \\ x_3 + x_4 + 2x_6 & = & 2 \\ x_5 - 2x_6 & = & 1 \\ 0 & = & 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} x_2 & = & 3 - 2x_4 - 3x_6 \\ x_3 & = & 2 - x_4 - 2x_6 \\ x_5 & = & 1 + 2x_6 \end{array}\right.$$

Step 4 : Write Down Solution as a Point, Line, Plane, or Hyperplane of Vectors

Formulate a solution vector using the equations for the fixed variables and leaving the free variable components alone and expand solution vector in terms of the free variables

$$\begin{aligned} x_1 &= \textit{free} \\ x_2 &= 3 - 2x_4 - 3x_6 \\ x_3 &= 2 - x_4 - 2x_6 \\ x_4 &= \textit{free} \\ x_5 &= 1 + 2x_6 \\ x_6 &= \textit{free} \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 3 - 2x_4 - 3x_6 \\ 2 - x_4 - 2x_6 \\ x_4 \\ 1 + 2x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -3 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Example: A Linear System with a Redundancy

Consider

$$\begin{aligned} x_1 + x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 3 \\ x_1 + x_2 + 2x_4 &= 4 \end{aligned}$$

Note Eq3 = Eq1 + Eq2, so Eq3 is already implied by Eq1 and Eq1; it thus amounts to a redundant condition on x_1, x_2, x_3, x_4.

Example, Cont'd

Step 1: Convert to Augmented Matrix

$$\left.\begin{array}{rcl} x_1 + x_3 + x_4 & = & 1 \\ x_2 - x_3 + x_4 & = & 3 \\ x_1 + x_2 + 2x_4 & = & 4 \end{array}\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 1 & 1 & 0 & 2 & 4 \end{array}\right]$$

Example with Redundancy, Cont'd

Step 2: Row Reduce to R.R.E.F.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 1 & 1 & 0 & 2 & 4 \end{array}\right] \xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (\textit{REF} \text{ and } \textit{RREF})$$

Example with Redundancy, Cont'd

Step 3: Solution as Equations
Note: x_3 and x_4 will be free variables in the solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{\begin{array}{ccc} x_1 + x_3 + x_4 & = & 1 \\ x_2 - x_3 + x_4 & = & 3 \\ 0 & = & 0 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{ccc} x_1 & = & 1 - x_3 - x_4 \\ x_2 & = & 3 + x_3 - x_4 \end{array}\right\}$$

Example with Redundancy, Cont'd

Step 4: Expand Solution Vector w.r.t. the Free Variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_3 - x_4 \\ 3 + x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is thus a 2-dimensional plane inside $\mathbb{R}^4$.

A Linear System with a Contradiction

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ x_1 - x_2 - x_3 + x_4 &= 2 \\ 2x_1 + 2x_4 &= 2 \end{aligned}$$

Note that the sum of the first two equations is

$$2x_1 + 2x_4 = 4$$

which contradicts the last equation. So there cannot be a valid solution to this system.

Example with a Contradiction, Cont'd

Row Reduction:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 1 & 2 \\ 2 & 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow[\mathcal{R}_3 \to \mathcal{R}_3 - 2\mathcal{R}_1]{\mathcal{R}_2 \to \mathcal{R}_2 + \mathcal{R}_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_3 \to \mathcal{R}_3 - \mathcal{R}_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Example with Contradiction: Row Echelon Form

$$R.E.F. \qquad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Last row says

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = -2$$

or

$$0 = -2 \qquad \text{(a contradiction)}$$

We conclude: **there is no solution to the original linear system**.