Lecture 10 : Inverses of Square Matrices

Math 3013
Oklahoma State University

February 4, 2022

Agenda:

1. Example: Solving Linear Systems
2. Inverses of Matrices
3. Elementary Matrices
4. Calculating the Inverse of a Matrix

Example: Solving Linear Systems

Express the solution of the following linear system as a hyperplane.

$$\begin{aligned} x_1 + x_2 + 2x_4 &= 1 \\ 2x_1 + 2x_2 + x_3 + 5x_4 &= 4 \\ x_3 + x_4 &= 2 \end{aligned}$$

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 5 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_1 + x_2 + 2x_4 &= 1 \\ x_3 + x_4 &= 2 \\ 0 &= 0 \end{aligned}\right\} \to \left\{\begin{aligned} x_1 &= 1 - x_2 - 2x_4 \\ x_3 &= 2 - x_4 \\ 0 &= 0 \end{aligned}\right.$$

Example, Cont'd

Solution equations expressing fixed variables in terms of free variables

$$\begin{aligned} x_1 &= 1 - x_2 - 2x_4 \\ x_3 &= 2 - x_4 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_2 - 2x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Inverses of Matrices

Motivation:

Consider an $n \times n$ linear system written in matrix notation:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Suppose there was an $n \times n$ matrix $\mathbf{C}$ such that

$$\mathbf{C}\mathbf{A} = \mathbf{I} \quad , \qquad \text{the } n \times n \text{ identity matrix}$$

Then multiplying both sides from the left by $\mathbf{C}$, we find

$$\begin{aligned} \mathbf{C}(\mathbf{A}\mathbf{x}) &= \mathbf{C}(\mathbf{b}) \\ &\Rightarrow \quad (\mathbf{C}\mathbf{A})\,\mathbf{x} = \mathbf{C}\mathbf{b} \\ &\Rightarrow \quad (\mathbf{I})\,\mathbf{x} = \mathbf{C}\mathbf{b} \\ &\Rightarrow \quad \mathbf{x} = \mathbf{C}\mathbf{b} \end{aligned}$$

In other words, if we had a matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$, then we could solve $\mathbf{Ax} = \mathbf{b}$ simply by multiplying both sides from the left by $\mathbf{C}$.

Definition

Suppose $\mathbf{A}$ is an $n \times n$ matrix. Then any matrix $\mathbf{C}$ such that

$$\mathbf{CA} = \mathbf{I} \quad \text{and} \quad \mathbf{AC} = \mathbf{I}$$

is called a **matrix inverse** of $\mathbf{A}$.

Example: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ Then $\mathbf{C} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ is a matrix inverse of $\mathbf{A}$; For

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3-2 & -2+2 \\ 3-3 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

and

$$\mathbf{CA} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3-2 & 6-6 \\ 1-1 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Properties of Matrix Inverses

- Only square ($n \times n$) matrices can have inverses.
- If a matrix $\mathbf{A}$ has an inverse then it is unique.
  Proof: Suppose

$$\begin{aligned} \mathbf{AC} &= \mathbf{I} = \mathbf{CA} \\ \mathbf{AD} &= \mathbf{I} = \mathbf{DA} \end{aligned}$$

Then, on the one hand,

$$\mathbf{DAC} = \mathbf{D}(\mathbf{AC}) = \mathbf{D}(\mathbf{I}) = \mathbf{D}$$

while, on the other,

$$\mathbf{DAC} = (\mathbf{DA})\mathbf{C} = (\mathbf{I})\mathbf{C} = \mathbf{C}$$

and so

$$\mathbf{D} = \mathbf{DAC} = \mathbf{C} \qquad \Rightarrow \qquad \mathbf{D} = \mathbf{C}$$

So the two inverses of $\mathbf{A}$ have to be the same matrix.

Properties of Inverse Matrices, Cont'd

Definition

The unique matrix inverse to $\mathbf{A}$, if it exists, is denoted by $\mathbf{A}^{-1}$

- If $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ matrices with inverses $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, respectively, then the product matrix $\mathbf{AB}$ has an inverse and it is

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Proof:

$$\begin{aligned}(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) &= \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1} \\ &= \mathbf{A}\left(\mathbf{I}\right)\mathbf{A}^{-1} \\ &= \mathbf{AA}^{-1} \\ &= \mathbf{I}\end{aligned}$$

So $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the unique inverse of the matrix $\mathbf{AB}$

Digression: Elementary Matrices

Definition

Let $\mathbf{I}$ be the $n \times n$ identity matrix and let $\mathcal{R}$ be an elementary row operations. Then the **elementary matrix** corresponding to $\mathcal{R}$ is the matrix $\mathbf{E}_{\mathcal{R}}$ obtained by applying the operation $\mathcal{R}$ to $\mathbf{I}$

$$\mathbf{E}_{\mathcal{R}} \equiv \mathcal{R}(\mathbf{I})$$

Examples of Elementary Matrices

Let $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the 2×2 identity matrix.

$$
\begin{aligned}
\mathbf{E}_{R_1 \leftrightarrow R_2} &= \mathcal{R}_{R_1 \leftrightarrow R_2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\mathbf{E}_{R_2 \to 3R_2} &= \mathcal{R}_{R_2 \to 3R_2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \\
\mathbf{E}_{R_2 \leftrightarrow R_2 + 2R_1} &= \mathcal{R}_{R_2 \to R_2 + 2R_1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}
\end{aligned}
$$

Elementary Matrices and Row Operations

Theorem

Suppose $\mathbf{E}_{\mathcal{R}}$ *is the* $n \times n$ *elementary matrix corresponding to a elementary row operation* $\mathcal{R}$. *Then for any* $n \times n$ *matrix* $\mathbf{A}$

$$\mathcal{R}(\mathbf{A}) = \mathbf{E}_{\mathcal{R}}\mathbf{A}$$

Thus, an elementary row operation can be implemented either directly on $\mathbf{A}$ or by multiplying $\mathbf{A}$ by the corresponding elementary matrix.

Examples: Implementing Row Operations via Multiplication by Elementary Matrices

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We can implement elementary row operations on $\mathbf{A}$ two different ways:

Case 1: Row Interchanges:

$$\mathcal{R}_{R_1 \leftrightarrow R_2}(\mathbf{A}) = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad , \quad \mathbf{E}_{R_1 \leftrightarrow R_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{E}_{\mathcal{R}_1 \leftrightarrow R_2}\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 + c & 0 + d \\ a + 0 & b + 0 \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

So

$$\mathcal{R}_{R_1 \leftrightarrow R_2}(\mathbf{A}) = \mathbf{E}_{R_1 \leftrightarrow R_2}\mathbf{A}$$

Examples: Implementing Row Operations via Multiplication by Elementary Matrices, Cont'd

Case 2: Row Rescalings:

$$\mathcal{R}_{R_2 \to 3R_2}(\mathbf{A}) = \begin{bmatrix} a & b \\ 3c & 3d \end{bmatrix} \quad , \quad \mathbf{E}_{R_2 \to 3R_2} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{E}_{\mathcal{R}_2 \to 3R_2}\mathbf{A} &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a+0 & b+0 \\ 0+3c & 0+3d \end{bmatrix} = \begin{bmatrix} a & b \\ 3c & 3d \end{bmatrix} \end{aligned}$$

So

$$\mathcal{R}_{R_2 \to 3R_2}(\mathbf{A}) = \mathbf{E}_{R_2 \to 3R_2}\mathbf{A}$$

Examples: Implementing Row Operations via Multiplication by Elementary Matrices, Cont'd

Case 3: Replacing a row by its sum with a multiple of another row:

$$\begin{aligned}
\mathcal{R}_{R_2 \longleftrightarrow R_2+2R_1}(\mathbf{A}) &= \begin{bmatrix} a & b \\ c+2a & d+2b \end{bmatrix} \\
\mathbf{E}_{R_2 \to R_2+2R_1} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\
\mathbf{E}_{R_2 \to R_2+2R_1}\mathbf{A} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a+0 & b+0 \\ 2a+c & 2b+d \end{bmatrix} = \begin{bmatrix} a & b \\ c+2a & d+2b \end{bmatrix}
\end{aligned}$$

So

$$\mathcal{R}_{R_1 \longleftrightarrow 2R_2}(\mathbf{A}) = \mathbf{E}_{R_2 \to R_2+2R_1}\mathbf{A}$$

Calculating Matrix Inverses

- Suppose a matrix $\mathbf{A}$ can be row reduced to the identity matrix.
- Then there is a sequence $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ that convert $\mathbf{A}$ to the identity matrix:

$$\mathbf{I} = \mathcal{R}_k\left(\cdots \mathcal{R}_2\left(\mathcal{R}_1\left(\mathbf{A}\right)\right)\right)$$

- Then there is a product of elementary matrices that does the same thing

$$\mathbf{I} = \mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}}\cdots\mathbf{E}_{\mathcal{R}_1}\mathbf{A}$$

- Since $\mathbf{I} = \mathbf{BA} \quad \Longrightarrow \quad \mathbf{B} = \mathbf{A}^{-1}$, we conclude

$$\mathbf{A}^{-1} = \mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}}\cdots\mathbf{E}_{\mathcal{R}_1}$$

Procedure for Calculating $\mathbf{A}^{-1}$

1. Find a sequence of elementary row operations $\mathcal{R}_1, \ldots, \mathcal{R}_k$ that convert $\mathbf{A}$ to the identity matrix.
2. $\mathbf{A}^{-1}$ will be the corresponding product of elementary matrices

$$\mathbf{A}^{-1} = \mathbf{E}_{\mathcal{R}_k} \mathbf{E}_{\mathcal{R}_{k-1}} \cdots \mathbf{E}_{\mathcal{R}_1}$$

Next: We'll make this method of computing $\mathbf{A}^{-1}$ more efficient.

A Row Reduction Algorithm for Computing $\mathbf{A}^{-1}$

Now let $[\mathbf{A} \mid \mathbf{I}]$ be the $n \times 2n$ matrix obtained by adjoining the $n \times n$ identity matrix to $\mathbf{A}$:

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{array}\right]$$

Suppose $\mathbf{A}$ can be row-reduced to the identity matrix via a sequence $\mathcal{R}_1, \ldots, \mathcal{R}_k$ of elementary row operations:

$$\mathcal{R}_k\left(\cdots \mathcal{R}_1\left(\mathbf{A}\right)\right) = \mathbf{I}$$

then under the same sequence of elementary row operations $[\mathbf{A} \mid \mathbf{I}]$ row reduces to

$$\begin{aligned} \mathcal{R}_k\left(\cdots \mathcal{R}_1\left([\mathbf{A} \mid \mathbf{I}]\right)\right) &= \left[\mathcal{R}_k\left(\cdots \mathcal{R}_1\left(\mathbf{A}\right)\right) \mid \mathcal{R}_k\left(\cdots \mathcal{R}_1\left(\mathbf{I}\right)\right)\right] \\ &= \left[\mathbf{I} \mid \mathcal{R}_k\left(\cdots \mathcal{R}_1\left(\mathbf{I}\right)\right)\right] \end{aligned}$$

A Row Reduction Algorithm for Computing $\mathbf{A}^{-1}$, Cont'd

So if $\mathbf{A}$ can be row-reduced to $\mathbf{I}$ via elementary row operations $\mathcal{R}_1, \ldots, \mathcal{R}_k$, we also have

$$\mathcal{R}_k\left(\cdots \mathcal{R}_1\left([\mathbf{A} \mid \mathbf{I}]\right)\right) = \left[\mathbf{I} \mid \mathcal{R}_k\left(\cdots \mathcal{R}_1\left(\mathbf{I}\right)\right)\right]$$

But

$$\mathcal{R}_k\left(\cdots \mathcal{R}_1\left(\mathbf{I}\right)\right) = \mathbf{E}_{\mathcal{R}_k} \cdots \mathbf{E}_{\mathcal{R}_1}\mathbf{I} = \mathbf{E}_{\mathcal{R}_k} \cdots \mathbf{E}_{\mathcal{R}_1} = \mathbf{A}^{-1}$$

We can now conclude:
If $\mathbf{A}$ row reduces to $\mathbf{I}$, then $[\mathbf{A} \mid \mathbf{I}]$ row reduces to $\left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$ using the same sequence of row operations.

Algorithm for Calculating $\mathbf{A}^{-1}$

We now note that $\left[\mathbf{I}|\mathbf{A}^{-1}\right]$ is always the Reduced Row Echelon Form of $[\mathbf{A}|\mathbf{I}]$:

$$\left[\mathbf{I}|\mathbf{A}^{-1}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & * & * & \cdots & * \\ 0 & 1 & \cdots & 0 & * & * & \cdots & * \\ \vdots & & \ddots & \vdots & * & * & \cdots & * \\ 0 & 0 & \cdots & 1 & * & * & \cdots & * \end{array}\right]$$

Calculating Matrix Inverses via Row Reduction

We have thus demonstrated

Theorem

Suppose $\mathbf{A}$ *is an* $n \times n$ *matrix*

- *If* $\mathbf{A}$ *is row reducible to the identity matrix, then the R.R.E.F. of* $[\mathbf{A} \mid \mathbf{I}]$ *is* $\left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$
- *Otherwise,* $\mathbf{A}$ *has no inverse*

So a method for calculating $\mathbf{A}^{-1}$ would be to row reduce $[\mathbf{A} \mid \mathbf{I}]$ to its R.R.E.F. $[\mathbf{A}' \mid \mathbf{I}']$

- If $\mathbf{A}' = \mathbf{I}$ then $\mathbf{A}^{-1} = \mathbf{I}'$
- Otherwise, $\mathbf{A}$ doesn't have an inverse.

Example

Example If possible, find the inverse of $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad (R.R.E.F. \text{ of } [\mathbf{A} \mid \mathbf{I}])$$

Since the left hand side of the R.R.E.F. of $[\mathbf{A} \mid \mathbf{I}]$ is the identity matrix, the right hand side will be $\mathbf{A}^{-1}$. Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example If possible, find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 5 & 2 \end{bmatrix}$

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

We can stop here; for the left hand side has a zero row - which implies it can not be reduced further to the identity matrix. This means $[\mathbf{A} \mid \mathbf{I}]$ **can not** be row reduced to $\left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$, and so $\mathbf{A}$ has no inverse.