Lecture 11 : The Fundamental Theorem of Invertible Matrices

Math 3013
Oklahoma State University

February 9, 2022

Agenda:

1. Matrix Inverses
2. The Calculation of a Matrix Inverse
3. The Fundamental Theorem of Invertible Matrices

Matrix Inverses

Definition

Suppose $\mathbf{A}$ is an $n \times n$ matrix. Then any matrix $\mathbf{B}$ such that

$$\mathbf{BA} = \mathbf{I} \quad \text{and} \quad \mathbf{AB} = \mathbf{I}$$

is called a **matrix inverse** of $\mathbf{A}$.

Theorem

If a square matrix $\mathbf{A}$ *has an inverse, it is unique.*

Notation: If $\mathbf{A}$ is invertible, we will denote its (unique) inverse by $\mathbf{A}^{-1}$

Calculating Matrix Inverses - the underlying idea

- Uniqueness of $\mathbf{A}^{-1}$:
  If we can find a matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$, then $\mathbf{C} = \mathbf{A}^{-1}$
- **Elementary matrices** allow one to implement elementary row operations via matrix multiplication

$$\text{If } \mathbf{E}_{\mathcal{R}} := \mathcal{R}\left(\mathbf{I}\right), \text{ then } \mathcal{R}\left(\mathbf{A}\right) = \mathbf{E}_{\mathcal{R}}\mathbf{A}$$

- If $\mathbf{A}$ can be row reduced to $\mathbf{I}$, there there will be a sequence of elemementary row operations $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ that transform $\mathbf{A}$ to $\mathbf{I}$:

$$\begin{aligned} \mathbf{I} &= \mathcal{R}_k\left(\cdots \mathcal{R}_2\left(\mathcal{R}_1\left(\mathbf{A}\right)\right)\right) \\ &= \mathbf{E}_{\mathcal{R}_k}\cdots\mathbf{E}_{\mathcal{R}_2}\mathbf{E}_{\mathcal{R}_1}\mathbf{A} \end{aligned}$$

$$\Rightarrow \qquad \mathbf{A}^{-1} = \mathbf{E}_{\mathcal{R}_k}\cdots\mathbf{E}_{\mathcal{R}_2}\mathbf{E}_{\mathcal{R}_1}$$

So all we need a procedure for directly calculating

$$\mathbf{A}^{-1} = \mathbf{E}_{\mathcal{R}_k}\cdots\mathbf{E}_{\mathcal{R}_2}\mathbf{E}_{\mathcal{R}_1}$$

Algorithm for Calculating $\mathbf{A}^{-1}$

Theorem

Suppose $\mathbf{A}$ *is an* $n \times n$ *matrix and let* $[\mathbf{A}''|\mathbf{I}''] = R.R.E.F.([\mathbf{A}|\mathbf{I}])$.

- *If* $\mathbf{A}'' = \mathbf{I}$, *then* $\mathbf{A}^{-1} = \mathbf{I}''$.
- *If* $\mathbf{A}'' \neq \mathbf{I}$, *then* $\mathbf{A}$ *has no inverse.*

Calculating Inverse Matrices: Example

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

We have

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow \qquad R_2 \to R_2 - \frac{3}{2}R_1$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow \qquad R_2 \to 2R_2$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow \qquad R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \end{array}\right] = R.E.F.([\mathbf{A}|\mathbf{I}])$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \end{array}\right]$$

$\downarrow \qquad R_3 \to -R_3$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$\downarrow \qquad R_2 + 2R_3$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$\downarrow \qquad R_1 \to R_1 - 4R_3$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -11 & 8 & 4 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$\downarrow \qquad R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -14 & 10 & 6 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -14 & 10 & 6 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$\downarrow \qquad R_1 \to \frac{1}{2} R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$R.R.E.F.([\mathbf{A}|\mathbf{I}]) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right] = \left[\mathbf{I}|\mathbf{A}^{-1}\right]$$

and so

$$\mathbf{A}^{-1} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$$

The Fundamental Theorem of Invertible Matrices

The following theorem shows the close connections between various problems we have considered.

Theorem

Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent (i.e., if any one of these statements is true, then all are true).

(a) $\mathbf{A}^{-1}$ *exists.*

(b) *Every linear system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

(c) $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$.

(d) *The R.R.E.F. of* $\mathbf{A}$ *is the* $n \times n$ *identity matrix.*

(e) $\mathbf{A}$ *is a product of elementary matrices.*

Proof: We'll demonstrate the following chain of implications

$$(a) \quad \Rightarrow \quad (b) \quad \Rightarrow \quad (c) \quad \Rightarrow \quad (d) \quad \Rightarrow \quad (e) \quad \Rightarrow \quad (a)$$

(a) $\Rightarrow$ (b)

To show : if $\mathbf{A}^{-1}$ exists, then $\mathbf{Ax} = \mathbf{b}$ has a unique solution

Suppose $\mathbf{A}$ is invertible, with inverse $\mathbf{A}^{-1}$.
I claim $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Indeed,

$$\mathbf{Ax} = \mathbf{A}\left(\mathbf{A}^{-1}\mathbf{b}\right) = \left(\mathbf{AA}^{-1}\right)\mathbf{b} = \mathbf{Ib} = \mathbf{b}$$

and so $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is a solution.
Now suppose $\mathbf{y}$ is any other solution of $\mathbf{Ax} = \mathbf{b}$. Then

$$\begin{aligned} \mathbf{Ay} = \mathbf{b} \quad & \Rightarrow \quad \mathbf{A}^{-1}\mathbf{Ay} = \mathbf{A}^{-1}\mathbf{b} \\ & \Rightarrow \quad \mathbf{Iy} = \mathbf{A}^{-1}\mathbf{b} \\ & \Rightarrow \quad \mathbf{y} = \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

(so $\mathbf{y}$ is the same solution as we had before.) $\square$

(b)⇒(c)

To show : if linear systems $\mathbf{Ax} = \mathbf{b}$ have unique solutions for each $\mathbf{b}$, then the only solution of $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Suppose every linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.
Then if we choose $\mathbf{b} = \mathbf{0}$, then $\mathbf{Ax} = \mathbf{0}$ has a unique solution.
On the other hand, $\mathbf{x} = \mathbf{0}$ is obviously a solution of $\mathbf{Ax} = \mathbf{0}$.
Since by hypothesis the solution of such an equation is unique, we conclude that $\mathbf{Ax} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as its unique solution.

□

(c) $\Rightarrow$ (d)

To show : if $\mathbf{x} = \mathbf{0}$ is the only solution of $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{A}$ is row reducible to the identity matrix

This follows from our row reduction algorithm for solving linear systems.
Assume $\mathbf{x} = \mathbf{0}$ is the unique solution to $\mathbf{Ax} = \mathbf{0}$
Then the solution equation for $\mathbf{x}$ corresponds to the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

for which the coefficient part (the "$\mathbf{A}$-part" of the augmented matrix) is the identity matrix $\mathbf{I}$.

(c) $\Rightarrow$ (d) , Cont'd

On the other hand, since all solutions to a linear system $\mathbf{Ax} = \mathbf{0}$ are obtainable by row reducing the augmented matrix $[\mathbf{A} \mid \mathbf{0}]$ to its unique Reduced Row Echelon Form, we can conclude that $\mathbf{A}$ must be row reducible to the identity matrix.

□

(d) $\Rightarrow$ (e)

To show : if $\mathbf{A}$ can be row reduced to $\mathbf{I}$, then $\mathbf{A}$ is a product of elementary matrices

Assume there is a sequence $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ of elementary row operations that systematically converts $\mathbf{A}$ to the identity matrix $\mathbf{I}$. Say,

$$\mathcal{R}_k\left(\mathcal{R}_{k-1}\left(\cdots\left(\mathcal{R}_1\left(\mathbf{A}\right)\right)\right)\right) = \mathbf{I}$$

or, implementing the row operations via matrix multiplication by elementary matrices,

$$\mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}}\cdots\mathbf{E}_{\mathcal{R}_1}\mathbf{A} = \mathbf{I}$$

This tells us that

$$\mathbf{A}^{-1} = \mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}}\cdots\mathbf{E}_{\mathcal{R}_1}$$

So $\mathbf{A}^{-1}$ is a product of elementary matrices

(d) $\Rightarrow$ (e), Cont'd

Next we use two facts

Lemma

If $\mathbf{E}_{\mathcal{R}}$ is an elementary matrix, then $\mathbf{E}_{\mathcal{R}}$ is invertible and $(\mathbf{E}_{\mathcal{R}})^{-1}$ is another elementary matrix.

Lemma

If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell$ are invertible matrices, then the matrix product $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_\ell$ is invertible and

$$(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_\ell)^{-1} = (\mathbf{A}_\ell)^{-1} \cdots (\mathbf{A}_2)^{-1} (\mathbf{A}_1)^{-1}$$

Thus,

$$\begin{aligned} \mathbf{A} &= \left(\mathbf{A}^{-1}\right)^{-1} \\ &= \left(\mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}} \cdots \mathbf{E}_{\mathcal{R}_1}\right)^{-1} \\ &= \mathbf{E}_{\mathcal{R}_1}^{-1}\mathbf{E}_{\mathcal{R}_2}^{-1} \cdots \mathbf{E}_{\mathcal{R}_k}^{-1} \end{aligned}$$

and so $\mathbf{A}$ is a product of elementary matrices.

(e) ⇒ (a)

To show: if **A** is a product of elementary matrices, then **A** is invertible.

Suppose

$$\mathbf{A} = \mathbf{E}_{\mathcal{R}_1} \cdots \mathbf{E}_{\mathcal{R}_k}$$

Then because each of the matrix factor $\mathbf{E}_{\mathcal{R}_i}$ is invertible, **A** is also invertible and, moreover,

$$\mathbf{A}^{-1} = \left(\mathbf{E}_{\mathcal{R}_1}\mathbf{E}_{\mathcal{R}_2} \cdots \mathbf{E}_{\mathcal{R}_k}\right)^{-1} = \left(\mathbf{E}_{\mathcal{R}_k}\right)^{-1} \cdots \left(\mathbf{E}_{\mathcal{R}_2}\right)^{-1}\left(\mathbf{E}_{\mathcal{R}_1}\right)^{-1}$$

Having completed the chain of implications

$$(a) \quad \Rightarrow \quad (b) \quad \Rightarrow \quad (c) \quad \Rightarrow \quad (d) \quad \Rightarrow \quad (e) \quad \Rightarrow \quad (a)$$

the theorem is now proved. □

Overview of Material to be Covered on First Exam

- Vectors as elements of $\mathbb{R}^n$
  - Vector addition
  - Scalar multiplication
  - The dot product
- Simple Geometric Contructs: Points, Lines, Planes and Hyperplanes as Sets of Vectors

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{p}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \cdots t_k\mathbf{v}_k; t_1, t_2, \ldots, t_k \in \mathbb{R}\}$$

- Matrices and Matrix Algebra
  - Matrix Addition
  - Scalar Multiplication of Matrices
  - Matrix Multiplication
  - Matrix Transposes

Overview, Cont'd

- Linear Systems
  - The geometry of solution sets of linear systems
    - solution set of single linear equation in m unknowns is a $(m-1)$-dimenional hyperplane in $\mathbb{R}^m$
    - solution set of n equations in m-unknowns is the intersection of n hyperplanes in $\mathbb{R}^m$
  - Naive Expection: Given n linear equations in m unknowns, one expects the solutions to form a $(m-n)$-dimensional hyperplane in $\mathbf{R}^m$
    Exceptions:
    - linear systems with redundant equations
    - linear systems with internal contradictions
  - Matrix formulation: $\mathbf{Ax} = \mathbf{b}$
  - Solving linear systems via row reduction
    - Augmented Matrices $[\mathbf{A} \mid \mathbf{b}]$
    - Row Echelon Form
    - Reduced Row Echelon Form
    - Fixed Variables and Free Variables
    - Solution Set as a Hyperplane

Overview, Cont'd

- Matrix Inverses
  - Calculating $\mathbf{A}^{-1}$
  - The Fundamental Theorem of Matrix Inverses

    Theorem: Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent

    (a) $\mathbf{A}$ is invertible.

    (b) Every linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

    (c) $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

    (d) The R.R.E.F. of $\mathbf{A}$ is the $n \times n$ identity matrix.

    (e) $\mathbf{A}$ is a product of elementary matrices.