Lecture 12 : Subspaces

Math 3013
Oklahoma State University

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Agenda

- Quick Overview of 1st Exam Topics
- The Span of a Set of Vectors
- Subspaces

Quick Overview of Material on First Midterm

I. Vectors in $\mathbb{R}^n$

A. Vector Addition $\quad + : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

B. Scalar Multiplication $\quad * : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

C. Linear Combinations of Vectors : e.g. $\alpha \mathbf{v} + \beta \mathbf{u} + \cdots$

D. Dot Product $\quad \cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

II. Geometry of Vector Spaces

A. Points, Lines, Planes and Hyperplanes

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{p}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \cdots t_k\mathbf{v}_k; t_1, t_2, \ldots, t_k \in \mathbb{R}\}$$

B. Solutions Sets of Linear Equations are (intersections of) Hyperplanes

$$a_1x_1 + \cdots + a_{m-1}x_{m-1} + a_mx_m = b$$

$$\Rightarrow \quad \mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{b}{a_m} \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ \vdots \\ 0 \\ -\frac{a_1}{a_m} \end{bmatrix} + \cdots + x_{m-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -\frac{a_{m-1}}{a_m} \end{bmatrix}$$

Overview, Cont'd

III. Matrices and Matrix Algebra

A. Matrices and Linear Systems : Augmented Matrices

$$\left.\begin{array}{ccc} a_{11}x_1 + \cdots + a_{1m}x_m & = & b_1 \\ \vdots \qquad\qquad \vdots & & \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m & = & b_n \end{array}\right\} \Longleftrightarrow \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right]$$

B. Matrix Multiplication

$$(\mathbf{AB})_{ij} = Row_i(\mathbf{A}) \cdot Col_j(\mathbf{B})$$

C. Matrix Addition and Scalar Multiplication

D. The Transpose of a Matrix

Overview, Cont'd

IV. Solving Systems of Linear Equations

A. Elementary Operations on Systems of Equations

B. Elementary Row Operations

(i) $R_i \longleftrightarrow R_j$

(ii) $R_i \longrightarrow \lambda R_i \quad , \quad \lambda \neq 0$

(iii) $R_i \longrightarrow R_i + \lambda R_j$

C. Row-Echelon Form
$$\begin{bmatrix} \underline{*} & * & \cdots & * & * \\ 0 & 0 & \underline{*} & \cdots & * \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \underline{*} & * \end{bmatrix}$$

D. Reduced Row-Echelon Form
$$\begin{bmatrix} \underline{1} & * & 0 & 0 & * \\ 0 & 0 & \underline{1} & 0 & * \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \underline{1} & * \end{bmatrix}$$

Overview, Cont'd

E. Solving Linear Systems

- convert to augmented matrix $[\mathbf{A} \mid \mathbf{b}]$
- row reduce to R.E.F. (pivots occur in down and to the right)
- row reduce further to R.R.E.F. (R.E.F. with pivots $= 1$ and 0's above and below pivots)
- identify fixed and free variables in solution
  - fixed variables $\longleftrightarrow$ columns of R.E.F. with pivots
  - free variables $\longleftrightarrow$ columns of R.E.F. without pivots
- write down solution set as a hyperplane
  - (i) R.R.E.F. $\longrightarrow$ equations expressing fixed variables in terms of free variables
  - (ii) Use equations from (ii) to express solution vectors in terms of the free variables
  - (iii) Expand the solution vector in terms of the free parameters

N.B. There is no solution whenever you have whose only non-zero entry is in the last column of the augmented matrix.

Overview, Cont'd

V. Inverses of Square Matrices
   A. Definition and Properties of Matrix Inverses
   B. Elementary Matrices
   C. Calculating Matrix Inverses
      - form adjoined matrix $[\mathbf{A} \mid \mathbf{I}]$
      - row reduce $[\mathbf{A} \mid \mathbf{I}]$ to R.R.E.F.
      - if L.H.S. of R.R.E.F matrix is the identity matrix $\mathbf{I}$, then the R.H.S. is $\mathbf{A}^{-1}$

        if not, $\mathbf{A}$ has no inverse
   D. Fundamental Theorem of Matrix inverses and $n \times n$ linear systems
      - $\mathbf{A}$ has an inverse.
      - $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$
      - the only solution of $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$
      - the R.R.E.F. of $\mathbf{A}$ is the identity matrix $\mathbf{I}$
      - $\mathbf{A}$ is a product of elementary matrices

Linear Combinations

Recall

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. A **linear combination** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an expression of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \qquad , \qquad c_1, \ldots, c_k \in \mathbb{R}$$

The span of a set of vectors

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. The **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the set of all possible linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$:

$$span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k) \equiv \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

Remark: Recall that a hyperplane in $\mathbb{R}^n$, is a set of the form

$$\mathcal{H} = \{\mathbf{p}_0 + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

So $span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a hyperplane for which $\mathbf{p}_0 = \mathbf{0}$ (the zero vector).
Thus, the span of a set of vectors is always a hyperplane that passes through the origin.

Closure under an operation

The span of a set of vectors is the first example of what we will call a **subspace** of $\mathbb{R}^n$.
However, to define subspaces in general, we need a couple more preliminary concepts.

Definition

A **operation** on a set S is just a procedure or function that can be applied to elements of that set.
A set S is **closed under an operation** f if $f(s)$ is an element of S for each $s \in S$.

Example

The vector space $\mathbb{R}^n$ is a set that is closed under both scalar multiplication and vector addition
This is because after applying scalar multiplication or vector addition to elements of $\mathbb{R}^n$, you just get back elements of $\mathbb{R}^n$:

$$\begin{aligned} \lambda &\in \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{R}^n &\Rightarrow \quad \lambda\mathbf{v} \in \mathbb{R}^n \\ \mathbf{v}_1, \mathbf{v}_2 &\in \mathbb{R}^n &\Rightarrow \quad \mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{R}^n \end{aligned}$$

Closure Under Scalar Multiplication and Vector Addition

Formalizing the notions of closure under scalar multiplication and vector addition:

Definition

A subset $S \subseteq \mathbb{R}^n$ is **closed under scalar multiplication** if

$$\lambda \in \mathbb{R} \text{ and } \mathbf{v} \in \mathbf{S} \quad \Longrightarrow \quad \lambda\mathbf{v} \in S$$

A subset $S \subseteq \mathbb{R}^n$ is **closed under vector addition** if

$$\mathbf{v}_1, \mathbf{v}_2 \in S \quad \Longrightarrow \quad (\mathbf{v}_1 + \mathbf{v}_2) \in S$$

Subspaces

Definition

A **subspace** of $\mathbb{R}^n$ is a subset S of $\mathbb{R}^n$ that is closed under both scalar multiplication and vector addition.

As the nomenclature suggests, **subspaces** can be thought of as smaller vector spaces sitting inside a larger vector space (like a subset is a smaller set sitting inside a larger set).

Howevever, **subsets of $\mathbb{R}^n$ are usually not closed under scalar multiplication and vector addition**.

Example: the unit circle

Let

$$S^1 = \{[x, y] \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

This set is not closed under scalar multiplication. E.g.,

$$\lambda = 2 \in \mathbb{R}\,, \quad \mathbf{v} = [1, 0] \in S^1 \quad \Rightarrow \quad \lambda\mathbf{v} = [2, 0] \notin S^1 \text{ since } 2^2 + 0^2 \neq 1$$

So S^1 is not closed under vector addition either. E.g.,

$$[1, 0]\,, [0, 1] \in S^1 \quad \Rightarrow \quad [1, 0] + [0, 1] = [1, 1] \notin S^1 \text{ since } 1^2 + 1^2 \neq 1$$

Since the unit circle S^1 is not closed under scalar multiplication and vector addition, S^1 **is not** a subspace.

Example: the set $T = \{[x, y] \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$

The subset T is closed under vector addition:
If $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, then $[n_1, n_2]$, $[m_1, m_2] \in T$. And then

$$\mathbf{v}_1 + \mathbf{v}_2 = [n_1 + m_1, n_2 + m_2]$$

Since the sum of two integers is always another integer, both components of the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ are integers.
Thus, $\mathbf{v}_1 + \mathbf{v}_1$ is always in T; and so, T is closed under vector addition.

Example: the set $T = \{[x, y] \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$, Cont'd

However, the set T is not closed under scalar multiplication.
To see this, choose $\lambda = \sqrt{2} \in \mathbb{R}$ and let $\mathbf{v} = [1, 0] \in T$. Then,

$$\lambda\mathbf{v} = \sqrt{2}\,[1, 0] = \left[\sqrt{2}, 0\right] \notin T \qquad \text{since } \sqrt{2} \text{ is not an integer}$$

Since $\lambda\mathbf{v} \notin T$, T is not closed under scalar multiplication.

Since T is not closed under **both** scalar multiplication and vector addition, T is not a subspace.

Example: A hyperplane in $\mathbb{R}^n$

Consider a hyperplane in $\mathbb{R}^n$

$$\mathcal{H} = \{\mathbf{p}_0 + s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

If $\mathcal{H}$ does not contain the zero vector $\mathbf{0}$ it is not a subspace.
To see this, suppose $\mathbf{v} \in \mathcal{H}$, and we choose $\lambda = 0$. Then

$$\lambda\mathbf{v} = 0\mathbf{v} = \mathbf{0} \notin \mathcal{H}$$

and so $\mathcal{H}$ is not closed under scalar multiplication and so $\mathcal{H}$ is not a subspace.

Example: the span of a set of vectors

Consider

$$S = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \{s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

Then if $\lambda \in \mathbb{R}$ and $\mathbf{v} = s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \in S$, then

$$\lambda\mathbf{v} = \lambda\,(s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k) = (\lambda s_1)\,\mathbf{w}_1 + \cdots + (\lambda s_k)\,\mathbf{w}_k$$

Since each of the scalar factors (λs_i) on the right is a real number, $\lambda\mathbf{v}$ is another element of S (for $\lambda\mathbf{v}$ is just another linear combination of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$).

So the span of a set of vectors is closed under scalar multiplication.

Example: the span of a set of vectors, Cont'd

$$S = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \{s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

Now choose two vectors in S:

$$\begin{aligned} \mathbf{v}_1 &= s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \\ \mathbf{v}_2 &= t_1\mathbf{w}_1 + \cdots + t_k\mathbf{w}_k \end{aligned}$$

Then

$$\mathbf{v}_1 + \mathbf{v}_2 = (s_1 + t_1)\,\mathbf{w}_1 + \cdots + (s_k + t_k)\,\mathbf{w}_k$$

Since the scalar factors $(s_i + t_i)$ on the right are all real numbers, $\mathbf{v}_1 + \mathbf{v}_2$ belongs to S. Hence, S is closed under vector addition.

Since S is closed under both scalar multiplication and vector addition, it is a subspace of $\mathbb{R}^n$.

The Solution Set of a Linear System

Let S be the solution set of an $n \times m$ linear system:

$$S = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{A}\mathbf{y} = \mathbf{b}\}$$

S is not closed under scalar multiplication: If $\mathbf{y}$ is a solution, then

$$\mathbf{A}(\lambda \mathbf{y}) = \lambda \mathbf{A}\mathbf{y} = \lambda \mathbf{b} \neq \mathbf{b}$$

so $\lambda \mathbf{y}$ is not a solution

S is not closed under vector addition: If $\mathbf{y}$ and $\mathbf{w}$ are solutions, then

$$\mathbf{A}(\mathbf{y} + \mathbf{w}) = \mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{w} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$$

so $\mathbf{y} + \mathbf{w}$ is not a solution.

Since the solution set is not closed under both scalar multiplication and vector addition, the solution set is a not a subspace of the vector space of variable values $\mathbb{R}^m$.

The Solution Set of a Homogeneous Linear System

A **homogeneous linear system** is a linear system of the form $\mathbf{Ax} = \mathbf{0}$ (where the right hand side is the zero vector).
Let S be the solution set of an $n \times m$ homogeneous linear system.

S is closed under scalar multiplication: If $\mathbf{y}$ is a solution, then

$$\mathbf{A}(\lambda\mathbf{y}) = \lambda\mathbf{Ay} = \lambda\mathbf{0} = \mathbf{0}$$

so $\lambda\mathbf{y}$ is also a solution

S is closed under vector addition: If $\mathbf{y}$ and $\mathbf{w}$ are solutions, then

$$\mathbf{A}(\mathbf{y} + \mathbf{w}) = \mathbf{Ay} + \mathbf{Aw} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $\mathbf{y} + \mathbf{w}$ is a solution.

Since the solution set is closed under both scalar multiplication and vector addition, the solution set is a subspace (of the vector space of variable values $\mathbb{R}^m$).

Remarks

- Hyperplanes $\mathcal{H} = \{\mathbf{p}_0 + s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$ are not subspaces in general, but
- Spanning sets $span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \{s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$ are always subspaces
- Solution sets of linear systems $\mathbf{Ax} = \mathbf{b}$ are not subspaces in general, but
- Solution sets of homogeneous linear systems $\mathbf{Ax} = \mathbf{0}$ are always subspaces.

In fact,

- Solution sets of linear systems $\mathbf{Ax} = \mathbf{b}$ correspond to hyperplanes and are not subspaces in general.
- Solution sets of homogeneous linear systems $\mathbf{Ax} = \mathbf{0}$ correspond to spanning sets and are always subspaces