Lecture 13 : Two Basic Prototypes for Subspaces

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Agenda

- Subspaces
- Counter-Examples of Subspaces
- Two Basic Prototypes for Subspaces
  - $span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) \equiv \{t_1\mathbf{w}_1 + \cdots + \mathbf{w}_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$
  - The Solution Set of a Homogeneous Linear System $\mathbf{Ax} = \mathbf{0}$

Closure Under Scalar Multiplication and Vector Addition

Definition

A subset $S \subseteq \mathbb{R}^n$ is **closed under scalar multiplication** if

$$\lambda \in \mathbb{R} \text{ and } \mathbf{v} \in S \qquad \Longrightarrow \qquad (\lambda \mathbf{v}) \in S$$

A subset $S \subseteq \mathbb{R}^n$ is **closed under vector addition** if

$$\mathbf{v}_1, \mathbf{v}_2 \in S \qquad \Longrightarrow \qquad (\mathbf{v}_1 + \mathbf{v}_2) \in S$$

Subspaces

Definition

A **subspace** of $\mathbb{R}^n$ is a subset S of $\mathbb{R}^n$ that is closed under both scalar multiplication and vector addition:

$$\begin{aligned} \lambda \in \mathbb{R} \text{ and } \mathbf{v} \in S \quad &\Longrightarrow \quad (\lambda \mathbf{v}) \in S \\ \mathbf{v}_1, \mathbf{v}_2 \in S \quad &\Longrightarrow \quad (\mathbf{v}_1 + \mathbf{v}_2) \in S \end{aligned}$$

As the nomenclature suggests, **subspaces** can be thought of as vector spaces sitting inside a larger vector space (like a subset is a smaller set sitting inside a larger set).

Example: the unit circle S^1

Let

$$S^1 = \{[x, y] \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

This set is **not closed** under scalar multiplication. E.g.,

$$\lambda = 2 \in \mathbb{R}\,, \quad \mathbf{v} = [1, 0] \in S^1 \quad \Rightarrow \quad \lambda\mathbf{v} = [2, 0] \notin S^1$$

since $2^2 + 0^2 \neq 1$
S^1 is **not closed** under vector addition either. E.g.,

$$[1, 0]\,, [0, 1] \in S^1 \quad \Rightarrow \quad [1, 0] + [0, 1] = [1, 1] \notin S^1$$

since $1^2 + 1^2 \neq 1$

Since the unit circle is not closed under scalar multiplication and vector addition, S_1 **is not a subspace**.

Example: the set $T = \{[x, y] \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$

The subset T is closed under vector addition: If $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, then $[n_1, n_2]$, $[m_1, m_2] \in T$. then

$$\mathbf{v}_1 + \mathbf{v}_2 = [n_1 + m_1, n_2 + m_2]$$

Since the sum of two integers is always another integer, both components of the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ are integers. Thus, $\mathbf{v}_1 + \mathbf{v}_1$ is always in T; and so, T is closed under vector addition.

Example: the set $T = \left\{[x, y] \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\right\}$, Cont'd

However, the set T is not closed under scalar multiplication.
To see this, choose $\lambda = \sqrt{2} \in \mathbb{R}$ and let $\mathbf{v} = [1, 0] \in T$. Then,

$$\lambda\mathbf{v} = \sqrt{2}\,[1, 0] = \left[\sqrt{2}, 0\right] \notin T \qquad \text{since } \sqrt{2} \text{ is not an integer}$$

Since $\lambda\mathbf{v} \notin T$, T is not closed under scalar multiplication.

Since T is not closed under **both** scalar multiplication and vector addition, T **is not a subspace**.

Example: A hyperplane in $\mathbb{R}^n$

Consider a hyperplane in $\mathbb{R}^n$

$$\mathcal{H} = \{\mathbf{p}_0 + s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

If $\mathcal{H}$ does not contain the zero vector $\mathbf{0}$ it is not a subspace. To see this, suppose $\mathbf{v} \in \mathcal{H}$, and we choose $\lambda = 0$. Then

$$\lambda\mathbf{v} = 0\mathbf{v} = \mathbf{0} \notin \mathcal{H}$$

and so $\mathcal{H}$ is not closed under scalar multiplication and so $\mathcal{H}$ is not a subspace.

A Simple Way to See $S \subset \mathbb{R}^n$ is not a subspace

The preceding counter-example generalizes as follows:

Lemma

Let S be a subset of $\mathbb{R}^n$. If $\mathbf{0} \notin S$, then S is not closed under scalar multiplication and hence S is not a subspace.

(scalar multiplication by 0 would always yield $\mathbf{0}$ and so would take you out of a set S that doesn't contain $\mathbf{0}$)

The Span of a Set of Vectors

Recall

Definition

Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a set of vectors in $\mathbb{R}^n$. The **span** of $\mathbf{w}_1, \ldots, \mathbf{w}_k$ is the set of all linear combinations of $\mathbf{w}_1, \ldots, \mathbf{w}_k$:

$$span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \{s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

First Basic Prototype of a Subspace: the span of a set of vectors

Let

$$S = span\left(\mathbf{w}_1, \ldots, \mathbf{w}_k\right) \equiv \{s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

Then if $\lambda \in \mathbb{R}$ and $\mathbf{v} = s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \in S$, then

$$\lambda\mathbf{v} = \lambda\left(s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k\right) = \left(\lambda s_1\right)\mathbf{w}_1 + \cdots + \left(\lambda s_k\right)\mathbf{w}_k$$

Since each of the scalar factors (λs_i) on the right is a real number, $\lambda\mathbf{v}$ is another element of S (for $\lambda\mathbf{v}$ is just another linear combination of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$).

So **the span of a set of vectors is always closed under scalar multiplication**.

Example: the span of a set of vectors, Cont'd

$$S = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \{s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

Now choose two vectors in S:

$$\begin{aligned} \mathbf{v}_1 &= s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \\ \mathbf{v}_2 &= t_1\mathbf{w}_1 + \cdots + t_k\mathbf{w}_k \end{aligned}$$

Then

$$\mathbf{v}_1 + \mathbf{v}_2 = (s_1 + t_1)\,\mathbf{w}_1 + \cdots + (s_k + t_k)\,\mathbf{w}_k$$

Since the scalar factors $(s_i + t_i)$ on the right are all real numbers, $\mathbf{v}_1 + \mathbf{v}_2$ belongs to S.

Hence, $S = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ **is closed under vector addition**.

Example: the span of a set of vectors, Cont'd

$\mathit{span}\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ is thus closed under both scalar multiplication and vector addition,

If $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subset \mathbb{R}^n$, **then**

$$\mathit{span}\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) \equiv \{t_1\mathbf{w}_1 + \cdots + t_k\mathbf{w}_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$$

is always a subspace of $\mathbb{R}^n$.

Solution Set of a Linear System

Let S be the solution set of an $n \times m$ linear system:

$$S = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{A}\mathbf{y} = \mathbf{b}\}$$

S is not closed under scalar multiplication: If $\mathbf{y}$ is a solution, then

$$\mathbf{A}(\lambda \mathbf{y}) = \lambda \mathbf{A}\mathbf{y} = \lambda \mathbf{b} \neq \mathbf{b}$$

so $\lambda \mathbf{y}$ is not a solution

S is not closed under vector addition: If $\mathbf{y}$ and $\mathbf{w}$ are solutions, then

$$\mathbf{A}(\mathbf{y} + \mathbf{w}) = \mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{w} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$$

so $\mathbf{y} + \mathbf{w}$ is not a solution.

Since the solution set is not automatically closed under either scalar multiplication and vector addition,
The solution set of a linear system is a not, in general, a subspace of the vector space of variable values $\mathbb{R}^m$.

The Solution Set of a Homogeneous Linear System

A **homogeneous linear system** is a linear system of the form $\mathbf{Ax} = \mathbf{0}$ (where the right hand side is the zero vector).

Let S be the solution set of an $n \times m$ homogeneous linear system.

S is closed under scalar multiplication: If $\mathbf{y}$ is a solution, then

$$\mathbf{A}(\lambda \mathbf{y}) = \lambda \mathbf{Ay} = \lambda \mathbf{0} = \mathbf{0}$$

so $\lambda \mathbf{y}$ is also a solution

S is closed under vector addition: If $\mathbf{y}$ and $\mathbf{w}$ are solutions, then

$$\mathbf{A}(\mathbf{y} + \mathbf{w}) = \mathbf{Ay} + \mathbf{Aw} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $\mathbf{y} + \mathbf{w}$ is a solution.

Since the solution set is closed under both scalar multiplication and vector addition, **the solution set of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is a subspace of the vector space of variable values $\mathbb{R}^m$.**

Remarks

- Hyperplanes $\mathcal{H} = \{\mathbf{p}_0 + s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$ are not subspaces in general , but
- Spanning sets $span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \{s_1\mathbf{w}_1 + \cdots + s_k\mathbf{w}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$ are subspaces

Similarly,

- Solution sets of linear systems $\mathbf{Ax} = \mathbf{b}$ are not subspaces in general, but
- Solution sets of homogeneous linear systems $\mathbf{Ax} = \mathbf{0}$ are subspaces.

Summary So Far

Recall

Definition

A subset S of a vector space $\mathbb{R}^n$ is a **subspace** if:

- $\mathbf{v} \in S$ and $\lambda \in \mathbb{R} \quad \Longrightarrow \quad \lambda\mathbf{v} \in S$
- $\mathbf{v}_1, \mathbf{v}_2 \in S \quad \Longrightarrow \quad \mathbf{v}_1 + \mathbf{v}_2 \in S$

Two basic prototypes for subspaces:

- the span of a set of vectors: e.g.,

$$\begin{aligned} S &= span(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) \\ &\equiv \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\} \end{aligned}$$

- the solution set of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$

We'll next focus on the first prototype.

Nomenclature

If

$$S = span(\mathbf{v}_1, \dots, \mathbf{v}_k)$$

we say that S is **the subspace generated** by the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$.

We also say that the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are a **set of generators** for S.

A key concern for us today is that **the vectors that generate a subspace are far from unique**.
In fact, even the number of generators is not unique for a given subspace.

Finding good coordinates for the points of a subspace

Since a subspace is (generally) a lower dimensional hyperplane inside a higher dimensional vector space $\mathbb{R}^n$, the usual coordinates for $\mathbb{R}^n$ are not really suitable coordinates for vectors in a subspace. Let's begin with an example:

Example 1.

Consider

$$S = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

For each choice of numbers c_1, c_2, c_3,

$$\mathbf{v}_{c_1,c_2,c_3} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

belongs to S.

Example, Cont'd

But the numbers c_1, c_2, c_3 do not provide good coordinates for the vectors in S since two different choices of c_1, c_2, c_3 can correspond to the same vector. E.g.,

$$\left.\begin{array}{l} c_1 = 1 \\ c_2 = 1 \\ c_3 = 0 \end{array}\right\} \quad \Rightarrow \quad (1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (0)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 2 \end{array}\right\} \quad \Rightarrow \quad (0)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (0)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Bases

Circumstances like the preceding example motivate the following definition:

Definition

A **basis** for a subspace S is a set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ such that **every** vector $\mathbf{v} \in S$ can be expressed as

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k$$

for exactly one choice of coefficients $c_1, c_2, \ldots, c_k$.

Remarks:

If $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ is a basis for a subspace S, then

- $S = span\,(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k)$
- If two linear combinations of the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ yield the same vector, then the coefficients have to be the same:

$$c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k = d_1\mathbf{b}_1 + \cdots + d_k\mathbf{b}_k$$

  requires

$$c_1 = d_1\ ,\ c_2 = d_2\ ,\ \ldots\ ,c_k = d_k$$

- Thus, to each vector in S there corresponds a unique list of numbers $[c_1, \ldots, c_k]$. These numbers provide suitable (i.e., unique) coordinates for each vector of S.

The Utility of Bases

If $B = \{\mathbf{w}_1, \ldots, \mathbf{w})k\}$ is a basis for a subspace $S \subset \mathbb{R}^n$,

- Each $\mathbf{v} \in S$ corresponds to a unique ordered list of k numbers
- To each element $\mathbb{R}^k$ corresponds to a unique element of S.

$$\mathbf{v} \in S \quad \Longrightarrow \quad \mathbf{v} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k \quad \Longleftrightarrow \quad [c_1, \ldots, c_k] \in \mathbb{R}^k$$