Lecture 15 : Subspaces, Bases, and Linear Independence

Math 3013
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Agenda

- Subspaces
- Coordinatization of Subspaces and Bases
- Linear Independence
- The Row Space of a Matrix

Recap

Definition

A **subspace** of $\mathbb{R}^n$ is a subset $S \subset \mathbb{R}^n$ such that

- whenever $\mathbf{v} \in S$ and $\lambda \in \mathbb{R}$, $\lambda\mathbf{v} \in S$
- whenever $\mathbf{v}_1, \mathbf{v}_2 \in S$, $\mathbf{v}_1 + \mathbf{v}_2 \in S$

Two Fundamental Ways Subspaces Arise

- The **span** of a set of vectors:

$$\begin{aligned} S &= span\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right) \\ &\equiv \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\} \end{aligned}$$

  Geometrically, this is a hyperplane that passes through the origin $\mathbf{0}$.

- The **solution set of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$**

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1m}x_m &= 0 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m &= 0 \end{aligned}$$

Problem: Finding good coordinates for the vectors lying in a subspace of $\mathbb{R}^n$

Basic idea: If $W = span(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, use coefficients of elements of W as coordinates

$$\begin{aligned} \mathbf{v} \in W \quad &\Longrightarrow \quad \mathbf{v} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k \\ &\Longrightarrow \quad \text{use } [c_1, \ldots, c_k] \text{ as coordinates for } \mathbf{v} \text{ in } W \end{aligned}$$

Problem: without some restrictions on the generators $\mathbf{w}_1, \ldots, \mathbf{w}_k$ such "coordinates" do not specify the vectors in W uniquely.

(Counter-)Example

Consider

$$S = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

For each choice of numbers c_1, c_2, c_3,

$$\mathbf{v}_{c_1,c_2,c_3} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

belongs to S.

Example, Cont'd

But the numbers c_1, c_2, c_3 do not provide good coordinates for the vectors in S since two different choices of c_1, c_2, c_3 can correspond to the same vector. E.g.,

$$\left.\begin{array}{l} c_1 = 1 \\ c_2 = 1 \\ c_3 = 0 \end{array}\right\} \quad \Rightarrow \quad (1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (0)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 2 \end{array}\right\} \quad \Rightarrow \quad (0)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (0)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Bases

Defining our way out of the problem:

Definition

A **basis** for a subspace S is a set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ such that **every** vector $\mathbf{v} \in S$ can be **uniquely** expressed as

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k$$

(i.e., the above equation is true for exactly one choice of coefficients $c_1, c_2, \ldots, c_k$)

Thus, an element $\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \in S$ is uniquely identified by its "coordinate vector" $[c_1, \ldots, c_k]$ **if and only if** $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for S.

Identifying Bases for Subspaces

Definition

A set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is **linearly independent** if the only solution of

$$x_1\mathbf{w}_1 + \cdots + x_k\mathbf{w}_k = \mathbf{0}$$

is

$$x_1 = 0\ ,\ x_2 = 0\ ,\ \ldots\ ,\ x_k = 0$$

Linear Independence and Bases

Theorem

Suppose $W = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$. *Then* $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ *is a basis for* W **if and only if** *the vectors* $\mathbf{w}_1, \ldots, \mathbf{w}_k$ *are linearly independent.*

Proof.

$\Longrightarrow$: To show : If $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis for W, then $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ are linearly independent.

Since B is a basis for W, every vector in W has a unique expression as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_k$.
In particular, since $\mathbf{0} \in W$,

$$\mathbf{0} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k$$

for exactly one choice of coeffients $c_1, \ldots, c_k$. But clearly, the choice $c_1 = 0, c_2 = 0, \ldots, c_k = 0$ works.
Thus,

$$x_1\mathbf{w}_1 + \cdots + x_k\mathbf{w}_k = \mathbf{0} \quad \Longrightarrow \quad x_1 = 0, x_2 = 0, \ldots, x_k = 0$$

and so $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are linear independent vectors.

Proof of Theorem, Cont'd

$\Longleftarrow$:

To show : If $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are linearly independent, then $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis for $W = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$.

Since $W = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, every vector $\mathbf{v} \in W$ is a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_k$. We need to show that there is only one way of writing a given $\mathbf{v} \in W$ as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_k$. Suppose there were two ways:

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k \\ \mathbf{v} &= d_1\mathbf{w}_1 + \cdots + d_k\mathbf{w}_k \end{aligned}$$

Subtracting the second equation from the first yields

$$\mathbf{0} = (c_1 - d_1)\,\mathbf{w}_1 + \cdots + (c_k - d_k)\,\mathbf{w}_k \qquad (*)$$

Since the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are linearly independent,

$$\mathbf{0} = x_1\mathbf{w}_1 + \ldots + x_k\mathbf{w}_k \quad \Longrightarrow \quad x_1 = 0, \ldots, x_k = 0$$

Thus, (*) requires

$$c_1 - d_1 = 0, \ldots, c_k - d_k = 0 \quad \Longrightarrow \quad c_1 = d_1, \ldots, c_k = d_k$$

Upshot:

To find a basis for a subspace W, we need to find a set of linearly independent vectors that generate W.

Put another way,
$\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for W **if and only if**

$$(i) \qquad W = span\,(\mathbf{b}_1, \ldots, \mathbf{b}_k)$$

$$(ii) \qquad \mathbf{0} = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \quad \Leftrightarrow \quad c_1 = 0, \ldots, c_k = 0$$

Nomenclature

Whenever one has an equation of the form

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \tag{1}$$

the solution $c_1 = 0, \ldots, c_k = 0$ of (1) is called the **trivial solution** of (1).
If there are other (non-trivial) solutions of (1), then (1) is called a **dependence relation**.

A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are **linearly dependent** if there exists a dependence relation for them.

Here is tale-tell sign of a set of linearly independent vectors.

Suppose each of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ has non-zero component that cannot be can not be cancelled out by a linear combination of the other vectors. Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Example: Consider the vectors $\mathbf{v}_1 = \{[1, 1, 0, 1]\ ,\ \mathbf{v}_2 = [0, 1, 1, 0]\}$. A dependence relation between these vectors would be an equation of the form

$$[0, 0, 0, 0] = x_1[1, 1, 0, 1] + x_2[0, 1, 1, 0] = [x_1, x_1 + x_2, x_2, x_1]$$

We must have $x_1 = 0$ because the first component of $\mathbf{v}_1$ cannot be cancelled by a scalar multiple of $\mathbf{v}_2$ and we must have $x_2 = 0$ since the third component of $\mathbf{v}_2$ cannot be cancelled by a scalar multiple of $\mathbf{v}_1$.

The Row Vectors of a Matrix in Row Echelon Form

The preceding observation implies

Theorem

Suppose **A** *is an* $n \times m$ *matrix in Row Echelon Form. Then the non-zero row vectors of* **A** *are linearly independent.*

Proof. If let $Row_1(\mathbf{A}), \ldots, Row_k(\mathbf{A})$ be the non-zero rows of **A**. Consider

$$x_1 Row_1(\mathbf{A}) + x_2 Row_2(\mathbf{A}) + \cdots + x_k Row_k(\mathbf{A}) = \mathbf{0} \qquad (*)$$

Since **A** is in R.E.F., below the pivot in the first row we'll have nothing but 0's; and so we can not cancel out the pivot component of $Row_1(\mathbf{A})$ using a linear combinations with the other row vectors. Hence, we cannot have (*) without $x_1 = 0$. But once the contribution of $Row_1(\mathbf{A})$ has been removed from (*), the same argument implies that

$$x_2 Row_2(\mathbf{A}) + \cdots + x_k Row_k(\mathbf{A}) = \mathbf{0} \quad \Longrightarrow \quad x_2 = 0$$

(since there is no way to cancel the pivot component of $\mathbf{R}_2(\mathbf{A})$)

The Row Vectors of a Matrix in R.E.F., Cont'd

Thus,

$$
\begin{aligned}
\mathbf{0} &= x_1 Row_1(\mathbf{A}) + x_2 Row_2(\mathbf{A}) + \cdots + x_k Row_k(\mathbf{A}) \\
&\qquad \Longrightarrow \quad x_1 = 0 \\
\Longrightarrow \quad \mathbf{0} &= x_2 Row_2(\mathbf{A}) + \cdots + x_k Row_k(\mathbf{A}) \quad \Longrightarrow \quad x_2 = 0 \\
&\vdots \\
\mathbf{0} &= x_k Row_k(\mathbf{A}) \quad \Longrightarrow \quad x_k = 0
\end{aligned}
$$

Thus, (*) can only hold when $x_1 = 0, \ldots, x_k = 0$. Hence, the non-zero row vectors of $\mathbf{A}$ are linearly independent. $\square$.

Finding a Basis for $RowSp(\mathbf{A})$

Definition

The **Row Space** of an $n \times m$ matrix is the subspace of $\mathbb{R}^m$ generated by the row vectors of $\mathbf{A}$.

$$RowSp(\mathbf{A}) = \{c_1 Row_1(\mathbf{A}) + \cdots + x_n Row_n(\mathbf{A}) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

Recall

Criteria for a Basis: $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for a subspace S if

- $S = span(\mathbf{b}_1, \ldots, \mathbf{b}_k)$
- $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ are linearly independent.

Corollary

If a matrix $\mathbf{A}$ *is in R.E.F., its non-zero row vectors provide a basis for* $RowSp(\mathbf{A})$.

What about Matrices that are not in R.E.F.?

Lemma

Suppose $\mathcal{R}$ is an elementary row operation. Then

$$RowSp(\mathcal{R}(\mathbf{A})) = RowSp(\mathbf{A})$$

Corollary

Suppose $\mathbf{A}$ is an $n \times m$ matrix and let $R.E.F.(\mathbf{A})$ be its row echelon form. Then the non-zero rows of $R.E.F.(\mathbf{A})$ form a basis for $RowSp(\mathbf{A})$.

Finding a basis for $W = span(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ when $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are not linearly independent

Procedure:

- Write the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ as the rows of a matrix $\mathbf{A}$
- Row reduce $\mathbf{A}$ to a matrix $\mathbf{A}'$ in R.E.F.
- The non-zero rows of $\mathbf{A}'$ will be a basis for $W = span(\mathbf{w}_1, \ldots, \mathbf{w}_k)$

This works because

- Elementary row operations do not change the row space of a matrix
- The non-zero rows of matrix in R.E.F. are linearly independent and generate its row space.

Example

Consider

$$W = span\left([1,0,1,1],[1,1,-1,0],[0,-1,2,1]\right)$$

Find a basis for W.

- Form a matrix $\mathbf{A}$ using the given vectors as rows

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

  Note

$$W = span\left([1,0,1,1],[1,1,-1,0],[0,-1,2,1]\right) = RowSp(\mathbf{A})$$

- Row reduce $\mathbf{A}$ to a matrix $\mathbf{A}'$ in R.E.F.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{A}'$$

Example, Cont'd

- The non-zero rows of $\mathbf{A}'$ will be a basis for $RowSp(\mathbf{A}) = W$. Thus, a suitable basis for W will be

$$B = \{[1, 0, 1, 1]\ ,\ [0, 1, -2, -1]\}$$