Lecture 16 : Subspaces Attached to a Matrix and their Bases

Math 3013
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Agenda

- Recap: Subspaces, Bases and Linear Independence
- Subspaces Attached to a Matrix
- Finding a Basis for $RowSp(\mathbf{A})$

Subspaces

Definition

A subset S of a vector space $\mathbb{R}^n$ is a **subspace** if:

- whenever $\mathbf{v} \in S$ and $\lambda \in \mathbb{R}$, $\lambda\mathbf{v} \in S$
- whenever $\mathbf{v}_1, \mathbf{v}_2 \in S$, $\mathbf{v}_1 + \mathbf{v}_2 \in S$

Two Fundamental Ways Subspaces Arise

(i) The **span** of a set of vectors:

$$\begin{aligned} S &= \mathit{span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_k) \\ &\equiv \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\} \end{aligned}$$

(ii) The **solution set of a homogeneous linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$**

Subspaces of $\mathbb{R}^n$: Geometric Picture

Geometrically, subspaces are hyperplanes that pass through the origin $\mathbf{0} \in \mathbb{R}^n$.

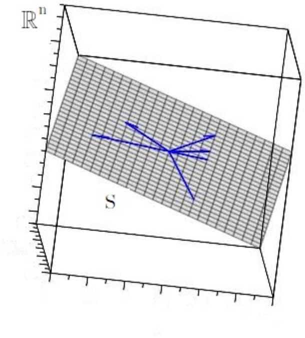

Figure: $S = span\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right) \subset \mathbb{R}^n$

Bases and Coordinatization of Subspaces

Problem: How to deal with n-dimensional vectors living in a k-dimensional subspace of $\mathbb{R}^n$?

Idea: If

$$S = span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k)$$

each $\mathbf{w} \in S$ can be expressed as

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

Use $[c_1, \ldots, c_k] \in \mathbb{R}^k$ as coordinates for $\mathbf{w}$.

However, if the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are not "linearly independent" the numbers $c_1, \ldots, c_k$ will not be not unique (leading to multiple coordinates for a given $\mathbf{w} \in S$).

Subspaces of $\mathbb{R}^n$: Geometric Picture (reprise)

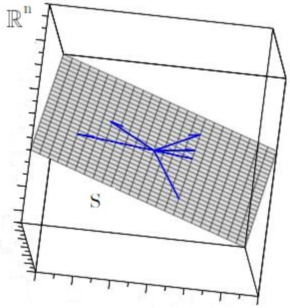

Figure: $S = span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k) \subset \mathbb{R}^n$

In general, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ will not be a basis for $S = span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

Linear Independence

Definition

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are said to be **linearly independent** if the only solution of

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

is

$$x_1 = 0\ ,\ x_2 = 0\ ,\ \dots\ ,\ x_k = 0$$

Theorem

Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and let $S = span\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Then

$$B \text{ is a basis for } S \quad \Longleftrightarrow \quad \text{The vectors in } B \text{ are linearly independent}$$

Revised Definition of Basis

Original Definition: A **basis** for a subspace S is a set of vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ such that every vector $\mathbf{w}$ in S can be **uniquely** written as

$$\mathbf{w} = c_1 \mathbf{b}_1 + \cdots + c_k \mathbf{b}_k$$

Since the numbers $c_1, \dots, c_k$ are unique, they provide "good coordinates" for $\mathbf{w} \in S$.
Revised Definition: A set of vectors $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is a **basis** for a subspace $S \subset \mathbb{R}^m$ if

(i) $S = span\,(\mathbf{b}_1, \dots, \mathbf{b}_k)$

(ii) $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ are linearly independent.

We then have

If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is a basis for a subspace $S \subset \mathbb{R}^n$, then every vector $\mathbf{w} \in S$ has a unique coordinate vector $\mathbf{w}_B \in \mathbb{R}^k$ defined by

$$\mathbf{w} = c_1 \mathbf{b}_1 + \cdots + c_k \mathbf{b}_k \quad \Longleftrightarrow \quad \mathbf{w}_B \equiv [c_1, \dots, c_k] \in \mathbb{R}^k$$

Geometric Picture: Basis Vectors in a Subspace

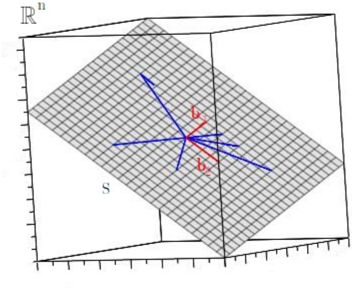

Figure: Basis $\{\mathbf{b}_1, \mathbf{b}_2\} \subset S = span(\mathbf{v}_1, \ldots, \mathbf{v}_k) \subset \mathbb{R}^n$

Digression: Subspaces attached to an $n \times m$ matrix

Definition

Let $\mathbf{A}$ be an $n \times m$ matrix. Attached to $\mathbf{A}$ are three natural subspaces:

(i) The **Row Space** of $\mathbf{A}$ is the span of the row vectors of $\mathbf{A}$

$$\begin{aligned} RowSp(\mathbf{A}) &\equiv span\left(Row_1(\mathbf{A}), Row_2(\mathbf{A}), \ldots, Row_n(\mathbf{A})\right) \\ &= \{c_1 Row_1(\mathbf{A}) + \cdots + c_n Row_n(\mathbf{A}) \mid c_1, \ldots, c_n \in \mathbb{R}\} \\ &\subset \mathbb{R}^m \end{aligned}$$

(ii) The **Column Space** of $\mathbf{A}$ is the span of the column vectors of $\mathbf{A}$

$$\begin{aligned} ColSp(\mathbf{A}) &\equiv span\left(Col_1(\mathbf{A}), Col_2(\mathbf{A}), \ldots, Col_m(\mathbf{A})\right) \\ &= \{c_1 Col_1(\mathbf{A}) + \cdots + c_m Col_m(\mathbf{A}) \mid c_1, \ldots, c_m \in \mathbb{R}\} \\ &\subset \mathbb{R}^n \end{aligned}$$

Subspaces attached to an $n \times m$ matrix, Cont'd

(iii) The **Null Space** of **A** is the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$:

$$\begin{aligned} NullSp(\mathbf{A}) &\equiv \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{Ax} = \mathbf{0}\} \\ &\subset \mathbb{R}^m \end{aligned}$$

Finding Bases for the Row Space of a Matrix

Lemma

If $\mathbf{A}$ is a matrix in R.E.F., then the non-zero row vectors of $\mathbf{A}$ are linearly independent and form a basis for $RowSp(\mathbf{A})$.

Idea of Proof:

Let $\mathbf{A}$ be a matrix in R.E.F. and let $\mathbf{r}_1, \ldots, \mathbf{r}_k$ be the non-zero rows of $\mathbf{A}$

Suppose

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_k\mathbf{r}_k = \mathbf{0} \tag{1}$$

The pivot entry of $\mathbf{r}_1$ cannot be cancelled with corresponding entries in other rows and so we must have $c_1 = 0$. Equation (1) then becomes

$$c_2\mathbf{r}_2 + \cdots + c_k\mathbf{r}_k = \mathbf{0} \tag{2}$$

But now the pivot entry of $\mathbf{r}_2$ cannot be cancelled with the corresponding entries in $\mathbf{r}_3, \ldots, \mathbf{r}_k$ and so $c_2 = 0$.

Repeating this argument, we eventually conclude that equation (1) implies $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.
Thus, the non-zero rows of a matrix in R.E.F. are necessarily linearly independent.

Since the nonzero rows of $\mathbf{A}$ are linearly independent and generate the row space of $\mathbf{A}$, the nonzero rows of $\mathbf{A}$ are a basis for $RowSp\,(\mathbf{A})$. $\square$

Lemma

Elementary row operations do not change the row space of a matrix.

Idea of Proof: Consider a matrix with 2 rows $\mathbf{r}_1$ and $\mathbf{r}_2$

$$\mathbf{A} = \begin{bmatrix} \leftarrow & \mathbf{r}_1 & \rightarrow \\ \leftarrow & \mathbf{r}_2 & \rightarrow \end{bmatrix} \Rightarrow \ RowSp\left(\mathbf{A}\right) = \{c_1\mathbf{r}_1 + c_2\mathbf{r}_2 \mid c_1, c_2 \in \mathbb{R}\}$$

(i) $\mathbf{A}' = \mathcal{R}_{R_1 \longleftrightarrow R_2}\left(\mathbf{A}\right)$

$$RowSp\left(\mathbf{A}'\right) = \{c_1\mathbf{r}_2 + c_2\mathbf{r}_1 \mid c_1, c_2 \in \mathbb{R}\} = RowSp\left(\mathbf{A}\right)$$

(ii) $\mathbf{A}'' = \mathcal{R}_{R_2 \rightarrow \lambda R_2}\left(\mathbf{A}\right)$

$$RowSp\left(\mathbf{A}''\right) = \{c_1\mathbf{r}_1 + c_2\lambda\mathbf{r}_2 \mid c_1, c_2 \in \mathbb{R}\} = RowSp\left(\mathbf{A}\right)$$

(iii) $\mathbf{A}''' = \mathcal{R}_{R_2 \rightarrow R_2 + \lambda R_1}\left(\mathbf{A}\right)$

$$\begin{aligned} RowSp\left(\mathbf{A}'''\right) &= \{c_1\mathbf{r}_1 + c_2\left(\mathbf{r}_2 + \lambda\mathbf{r}_1\right) \mid c_1, c_2 \in \mathbb{R}\} \\ &= \{\left(c_1 + \lambda c_2\right)\mathbf{r}_1 + c_2\mathbf{r}_2 \mid c_1, c_2 \in \mathbb{R}\} = RowSp\left(\mathbf{A}\right) \end{aligned}$$

Corollary

Suppose $\mathbf{A}$ *is an* $n \times m$ *matrix and let* $\mathbf{A}'$ *be any R.E.F. of* $\mathbf{A}$ *Then the non-zero rows of the R.E.F.* $(\mathbf{A}')$ *form a basis for* $RowSp\,(\mathbf{A})$.

Proof: From first lemma

$$RowSp\left(\mathbf{A}'\right) = RowSp\,(\mathbf{A})$$

and so, by the second lemma,

$$\begin{aligned} \text{non-zero rows of R.E.F. } \mathbf{A}' &= \text{basis for } RowSp\left(\mathbf{A}'\right) \\ &= \text{basis for } RowSp\,(\mathbf{A}) \end{aligned}$$

Application: Finding a basis for $S = span(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ when $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are not linearly independent

Procedure:

- Write the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ as the rows of a matrix $\mathbf{A}$
- Row reduce $\mathbf{A}$ to a matrix $\mathbf{A}'$ in R.E.F.
- The non-zero rows of $\mathbf{A}'$ will be a basis for $RowSp(\mathbf{A}') = RowSp(\mathbf{A}) = span(\mathbf{w}_1, \ldots, \mathbf{w}_k) = S$

This works because

- Elementary row operations do not change the row space of a matrix
- The non-zero rows of matrix in R.E.F. provide a basis for its row space.

Example

Consider

$$S = span\left([1, 0, 1, 1], [1, 1, -1, 0], [0, -1, 2, 1]\right)$$

Find a basis for S.

- Form a matrix $\mathbf{A}$ using the given vectors as rows

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

  Note

$$S = span\left([1, 0, 1, 1], [1, 1, -1, 0], [0, -1, 2, 1]\right) = RowSp(\mathbf{A})$$

Example, Cont'd

- Row reduce $\mathbf{A}$ to a matrix $\mathbf{A}'$ in R.E.F.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{A}'$$

- The non-zero rows of $\mathbf{A}'$ will be a basis for $RowSp(\mathbf{A}) = S$.
- Thus, a suitable basis for S will be

$$B = \{[1,0,1,1]\ ,\ [0,1,-2,-1]\}$$

Bases for Solution Sets of Homogeneous Linear Systems

Recall there are two basic prototypes for a subspace of $\mathbb{R}^m$

- the **span** of a set of vectors
- the **solution set** of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$

We have just described how to find a basis for the first type of subspace.

As it turns out, we already know how to find a basis for the solution set of a homogeneous linear system. For such bases are automatically produced by our method of solving linear systems.

Bases for Solution Sets of Homogeneous Linear Systems, Cont'd

Recall our method of solving linear systems $\mathbf{Ax} = \mathbf{0}$

- form augmented matrix $[\mathbf{A} \mid \mathbf{0}]$
- row reduce $[\mathbf{A} \mid \mathbf{0}]$ to its Reduced Row Echelon Form $[\mathbf{A}'' \mid \mathbf{0}]$ (Note that the last column remains all zeros throughout the row reduction)
- write down the equations corresponding to $[\mathbf{A}'' \mid \mathbf{0}]$ and move the free variables to the right hand side. These equations then express the fixed variables in terms of the free variables.
- write down a typical solution vector $\mathbf{x}$ and then expand that vector in terms of the free variables (Note there that will be no constant vector in the expansion.)

Bases for Solution Sets of Homogeneous Linear Systems, Cont'd

The solution set S of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is thus a subspace of the form

$$S = \{s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

(Here $s_1, \ldots, s_k$ are the free parameters of the solution.)

Theorem

Suppose

$$S = \{s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k \mid s_1, \ldots, s_k \in \mathbb{R}\}$$

is the solution set of a homogeneous linear system constructed as described above. Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *are a basis for* S

Example

Find a basis for the solution set of

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{array}{rcl} [\mathbf{A} \mid \mathbf{0}] & = & \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right] \\ & \downarrow & \text{row reduction} \\ [\mathbf{A}'' \mid \mathbf{0}] & = & \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right] \\ & \downarrow & \end{array}$$

$$\begin{array}{c} x_1 = -x_4 \\ x_2 = x_3 \end{array} \quad \Rightarrow \quad \mathbf{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Note how $\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad x_3 = 0$ and $x_4 = 0$

Example, Cont'd

Thus,

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are linearly independent.

Since these two vectors are linearly independent and generate the solution set S, they form a basis for the solution set.