Lecture 17 : Dimension of a Subspace and the Rank of a Matrix

Math 3013
Oklahoma State University

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Agenda

- Recap: Subspaces and Bases
- Finding Bases for $RowSp(\mathbf{A})$, $NullSp(\mathbf{A})$, and $ColSp(\mathbf{A})$
- Dimensions of Subspaces
- The Rank and Nullity of a Matrix

Recap: Subspaces

Definition

A subset S of a vector space $\mathbb{R}^n$ is a **subspace** if:

- $\mathbf{v} \in S$ and $\lambda \in \mathbb{R} \implies \lambda\mathbf{v} \in S$
- $\mathbf{v}_1, \mathbf{v}_2 \in S \implies \mathbf{v}_1 + \mathbf{v}_2 \in S$

Two Basic Prototypes for Subspaces

(i) The **span** of a set of vectors:

$$\begin{aligned} S &= span\left(\mathbf{v}_1, \dots, \mathbf{v}_k\right) \\ &\equiv \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\} \end{aligned}$$

(ii) The **solution set of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$**

Recap: Bases

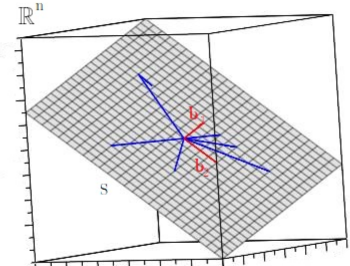
A basis for a subspace S is a special set of vectors that allows one to "coordinatize" S.

More precisely, if $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for a subspace $S \subset \mathbb{R}^n$, then

- for every $\mathbf{v} \in S$ there are **unique** numbers $c_1, \ldots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k$$

(The numbers $c_1, \ldots, c_k$ can then serve as coordinates of the vector $\mathbf{v}$ within S.)



Finding Bases for Subspaces: Case (i)

Suppose

$$W = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$$

Then a basis for W can be found as follows:

- Form a matrix $\mathbf{A}$ using the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ as rows.
- Row reduce $\mathbf{A}$ to a Row Echelon Form $\mathbf{A}'$
- The non-zero row vectors of the R.E.F. $\mathbf{A}'$ will be a basis for W

This was covered in the previous lecture.
However, I'll do examples of this procedure later in today's lecture.

Finding Bases for Subspaces: Case (ii)

Let $\mathbf{A}$ be an $n \times m$ matrix and let

$$W = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

be the solution set of the corresponding homogeneous linear system.

As I'll show in the next example:

Once we express the solution set W of $\mathbf{A}\mathbf{x} = \mathbf{0}$ as a hyperplane, we simultaneously obtain a basis for W.

Example

Find a basis for the solution set of

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{aligned}
[\mathbf{A} \mid \mathbf{0}] &= \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right] \\
\downarrow & \quad \text{row reduction} \\
[\mathbf{A}'' \mid \mathbf{0}] &= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right] = R.R.E.F.([\mathbf{A} \mid \mathbf{0}]) \\
\downarrow &
\end{aligned}$$

$$\begin{aligned} x_1 &= -x_4 \\ x_2 &= x_3 \end{aligned} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_4 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Note how $\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad x_3 = 0$ and $x_4 = 0$

Example, Cont'd

Thus,

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are linearly independent.

Since these two vectors are linearly independent and generate the solution set, they form a basis for the solution set.

Bases for the Subspaces Attached to an $n \times m$ Matrix $\mathbf{A}$

Let $\mathbf{A}$ be an $n \times m$ matrix. We have 3 associated subspaces

$$\begin{aligned} RowSp(\mathbf{A}) &= \text{span of the row vectors of } \mathbf{A} \\ ColSp(\mathbf{A}) &= \text{span of the column vectors of } \mathbf{A} \\ Null(\mathbf{A}) &= \text{solution set of } \mathbf{Ax} = \mathbf{0} \end{aligned}$$

- To find a basis for $RowSp(\mathbf{A})$, we simply row reduce $\mathbf{A}$ to a R.E.F. $\mathbf{A}'$ and then grab the non-zero rows of $\mathbf{A}'$
- To find a basis for $Null(\mathbf{A})$, we express the solutions of $\mathbf{Ax} = \mathbf{0}$ as a hyperplane and grab the constant vectors being multiplied by the free parameters of the solution.
- To find a basis for $ColSp(\mathbf{A})$, we can write the columns of $\mathbf{A}$ as the rows of a matrix, which will be the transpose $\mathbf{A}^t$ of $\mathbf{A}$. Then we row reduce $\mathbf{A}^t$ to a row echelon form $\mathbf{A}^{t\prime}$ and grab its non-zero rows, and convert back to column vectors

An alternative method for finding a basis for $ColSp(\mathbf{A})$

Theorem

If $\mathbf{A}$ is an $n \times m$ matrix, then a basis for $ColSp(\mathbf{A})$ can obtained as follows:

- *Row reduce $\mathbf{A}$ to a Row Echelon Form $\mathbf{A}'$*
- *Identify the columns of $\mathbf{A}'$ that contain pivots*
- *grab the corresponding columns of $\mathbf{A}$ (the original matrix)*

Finding bases for all 3 subspaces attached to a matrix $\mathbf{A}$ by row reducing $\mathbf{A}$ to its R.R.E.F.

Suppose $\mathbf{A}''$ is the R.R.E.F. of an $n \times m$ matrix $\mathbf{A}$:

- the nonzero rows of $\mathbf{A}''$ will be a basis for $RowSp(\mathbf{A})$
- the columns of $\mathbf{A}$ that correspond to the columns of $\mathbf{A}''$ that contain pivots will be a basis for $ColSp(\mathbf{A})$
- Use the R.R.E.F. of $\mathbf{A}$ to express the solutions of $\mathbf{Ax} = \mathbf{0}$ as a hyperplane, and then grab the constant vectors being multiplied by the free variables. These vectors will be a basis for $Null(\mathbf{A})$

Example

Find bases for the row space, the column space and the null space of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 0 & 2 \end{bmatrix}$$

We have

$$\begin{aligned} RREF\left(\mathbf{A}\right) &= \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \Rightarrow &\quad \text{basis for } RowSp\left(\mathbf{A}\right) = \{[1,0,1,1]\ ,\ [0,1,-2,-2]\} \\ \Rightarrow &\quad \text{basis for } ColSp\left(\mathbf{A}\right) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Example, Cont'd

From the R.R.E.F. of $\mathbf{A}$ we see that the solutions of $\mathbf{Ax} = \mathbf{0}$, will be of the form

$$\mathbf{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

and so $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ will be a basis for $Null\,(\mathbf{A})$

Dimension of a Subspace

Here are some fundamental facts about subspaces and their bases.

Theorem

Every subspace except $\{\mathbf{0}\}$ *has a basis.*

Theorem

If W *is a subspace, then every basis for* W *has the same number of vectors.*

These two theorems motivate the following definition of **dimension**.

Definition

The **dimension** of a subspace W is the common number of vectors in any basis for W.

Dimensions of Subspaces Attached to an $n \times m$ matrix $\mathbf{A}$

$$\begin{aligned}
\dim(RowSp(\mathbf{A})) &= \#\text{ basis vectors for } RowSp(\mathbf{A}) \\
&= \#\text{ non-zero rows in a R.E.F. of } \mathbf{A} \\
&= \#\text{ pivots in a R.E.F. of } \mathbf{A}
\end{aligned}$$

$$\begin{aligned}
\dim(ColSp(\mathbf{A})) &= \#\text{ basis vectors for } ColSp(\mathbf{A}) \\
&= \#\text{ pivots in a R.E.F. of } \mathbf{A}
\end{aligned}$$

Thus,

$$\dim(RowSp(\mathbf{A})) = \dim(ColSp(\mathbf{A}))$$

The Rank of a Matrix

Definition

Let $\mathbf{A}$ be an $n \times m$ matrix.
The common dimension of the subspaces $RowSp(\mathbf{A})$ and $ColSp(\mathbf{A})$ is called the **rank** of $\mathbf{A}$.

Note:

$$Rank(\mathbf{A}) = \begin{cases} \dim(RowSp(\mathbf{A})) \\ \dim(ColSp(\mathbf{A})) \\ \#\text{pivots in any R.E.F. of } \mathbf{A} \end{cases}$$

The Dimension of $NullSp(\mathbf{A})$

$$
\begin{aligned}
\dim(Null(\mathbf{A})) &= \#\text{ basis vectors for } NullSp(\mathbf{A}) \\
&= \#\text{ free parameters in solution of } \mathbf{Ax} = \mathbf{0} \\
&= \#\text{ columns without pivots in a R.E.F. of } \mathbf{A}
\end{aligned}
$$

Nomenclature: The text and WebAssign refer to $\dim(Null(\mathbf{A}))$ as the **nullity** of $\mathbf{A}$

Thus, one has

$$
\begin{aligned}
Nullity(\mathbf{A}) &= dim(NullSp(\mathbf{A})) \\
&= \#\text{ columns without pivots in any R.E.F. of } \mathbf{A}
\end{aligned}
$$

Theorem

Theorem

If **A** *is a matrix, then*

$$Rank\,(\mathbf{A}) + \dim\,(Null\,(\mathbf{A})) = \#Columns\ of\ \mathbf{A}$$

Proof: The rank of **A** is equal to the number of columns with pivots in a row echelon form of **A**.
$dim\,(Null\,(\mathbf{A}))$ is equal to the number of columns without pivots in a R.E.F. of **A**. Thus, adding these two numbers,

$$\begin{aligned} Rank\,(\mathbf{A}) + \dim\,(Null\,(\mathbf{A})) &= (\#\text{ columns with pivots}) \\ &\quad + (\#\text{columns without pivots}) \\ &= \#\text{ columns} \end{aligned}$$