Lecture 18: Linear Transformations

Math 3013
Oklahoma State University

March 4, 2022

Agenda

- Subspaces attached to a Matrix and their Bases: Example
- Functions Between Sets
- Linear Transformations

Subspaces attached to a Matrix and their Bases

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

and suppose $\mathbf{A}''$ is the R.R.E.F. of $\mathbf{A}$.

(i) The Row Space of $\mathbf{A}$

$$RowSp\left(\mathbf{A}\right) \equiv span\left(\left[a_{11}, \ldots, a_{1m}\right], \ldots, \left[a_{n1}, \ldots, a_{nm}\right]\right) \subset \mathbb{R}^m$$

The non-zero row vectors of $\mathbf{A}''$ form a basis for $RowSp\left(\mathbf{A}\right)$

Subspaces attached to a Matrix and their Bases, Cont'd

(ii) Column Space of **A**

$$ColSp(\mathbf{A}) = span\left(\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix}\right) \subset \mathbb{R}^n$$

The columns of **A** corresponding to the columns of the R.R.E.F. $\mathbf{A}''$ that contain pivots form a basis for $ColSp(\mathbf{A})$

(iii) The Null Space of **A**

$$NullSp(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{Ax} = \mathbf{0}\} \subset \mathbb{R}^m$$

When the solutions of $\mathbf{Ax} = \mathbf{0}$ are expressed as points on a hyperplane, the constant vectors being multiplied by the free variables form a basis for $NullSp(\mathbf{A})$

Dimensions of Subspaces

Definition

The **dimension** of a subspace W is the number of vectors in any basis for W.

(i) $dim\left(RowSp\left(\mathbf{A}\right)\right) = \#$ of pivots in any R.E.F. of $\mathbf{A}$.

(ii) $dim\left(ColSp\left(\mathbf{A}\right)\right) = \#$ of pivots in any R.E.F. of $\mathbf{A}$.

(iii) $dim\left(NullSp\left(\mathbf{A}\right)\right) = \#$ of columns w/o pivots any R.E.F. of $\mathbf{A}$.

Nomenclature

The **rank** of a matrix $\mathbf{A}$ is the common dimension of $RowSp\left(\mathbf{A}\right)$ and $ColSp\left(\mathbf{A}\right)$.

The **nullity** of a matrix $\mathbf{A}$ is the dimension of $NullSp\left(\mathbf{A}\right)$.

Note

$$\#\text{ columns of } \mathbf{A} = Rank\left(\mathbf{A}\right) + Nullity\left(\mathbf{A}\right)$$

Example

$$\text{Given } \mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = R.R.E.F.(\mathbf{A})$$

Deduce

(a) a basis for $RowSp(\mathbf{A})$

(b) a basis for $ColSp(\mathbf{A})$

(c) a basis for $NullSp(\mathbf{A})$

(d) the rank of $\mathbf{A}$

(e) the nullity of $\mathbf{A}$

(a) Basis for $RowSp(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = R.R.E.F.(\mathbf{A})$$

The non-zero rows of $R.R.E.F.(\mathbf{A})$ will provide a basis for $RowSp(\mathbf{A})$.

Thus,

$$\text{basis for } RowSp(\mathbf{A}) = \{[1, 0, 5] \ , \ [0, 1, -3]\}$$

(b) Basis for $ColSp\,(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = R.R.E.F.\,(\mathbf{A})$$

A basis for $ColSp\,(\mathbf{A})$ is given by the columns of $\mathbf{A}$ that correspond to the columns of $R.E.F.\,(\mathbf{A})$ that have pivots. Our matrix in R.R.E.F. has pivots in its first two columns; so we can use columns 1 and 2 of $\mathbf{A}$:

$$\text{basis for } ColSp\,(\mathbf{A}) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$$

(c) Basis for $NullSp\left(\mathbf{A}\right)$

A basis for $Null\left(\mathbf{A}\right)$ is given by the constant vectors that occur in the hyperplane form of the solution to $\mathbf{Ax} = \mathbf{0}$.
Now

$$R.R.E.F.\left([\mathbf{A}|\mathbf{0}]\right) = \left[R.R.E.F.\left(\mathbf{A}\right)|\mathbf{0}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so the corresponding equations of the solution are

$$\left.\begin{array}{r} x_1 + 5x_3 = 0 \\ x_2 - 3x_3 = 0 \end{array}\right\} \longrightarrow \left\{\begin{array}{l} x_1 = -5x_3 \\ x_2 = 3x_3 \end{array}\right.$$

$$\mathbf{x} = \begin{bmatrix} -5x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$$

The constant vector being multiplied by the free parameter x_3 will be the (only) basis vector for $NullSp\left(\mathbf{A}\right)$: Thus,

$$\text{basis for } Null\left(\mathbf{A}\right) = \left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Rank and Nullity of **A**

$$\begin{aligned} Rank\left(\mathbf{A}\right) &= \#\text{ pivots in } R.R.E.F.\left(\mathbf{A}\right) \\ &= 2 \end{aligned}$$

$$\begin{aligned} Nullity\left(\mathbf{A}\right) &= dim\left(NullSp\left(\mathbf{A}\right)\right) \\ &= \#\text{ columns of R.E.F. without pivots} \\ &= 1 \end{aligned}$$

Functions Between Sets

We will now turn to a new topic, the study of an simple class of functions between vector spaces.

We'll begin by recalling a little bit of the basic theory of functions between two sets.

Definition

A **function** $f : X \to Y$ from a set X to a set Y is a rule for each element of Y to an element of Y. The set X is called the **domain** of f, and the set Y is called the **codomain** of f.

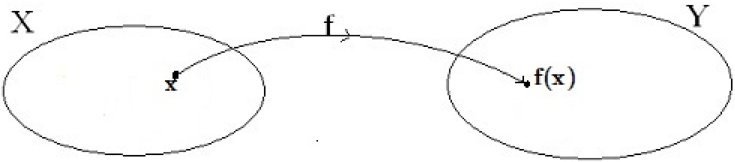

Figure: $f : X \to Y$

Functions Between Vector Spaces, Linear Transformations

We are going to be interested in a special family of functions between vector spaces; characterized by their behavior w.r.t. the two fundamental vector space operations; scalar multiplication and vector addition

Definition

A **linear transformation** is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

(i) $\lambda \in \mathbb{R}\ ,\ \mathbf{v} \in \mathbb{R}^m \quad \Rightarrow \quad T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$

(ii) $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^m \quad \Rightarrow \quad T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$

When I say that "linear transformations preserve scalar multiplication and vector addition", I am referring to properties (i) and (ii) above.

Linear Transformations, Cont'd

If a function $T : \mathbb{R}^m \to \mathbb{R}^n$ preserves both vector addition and scalar addition,
Then it will also preserve any combination of such operations;
Thus, it will preserve arbitrary linear combinations of vectors

and so

Lemma
If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then

$$\mathbf{T}(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1\mathbf{T}(\mathbf{v}_1) + r_2\mathbf{T}(\mathbf{v}_2) + \cdots + r_k\mathbf{T}(\mathbf{v}_k)$$

Linear Transformations and Linear Functions

Suppose we have a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Let $\mathbf{x} \in domain(T) = \mathbb{R}^m$, and let $\mathbf{y} = T(\mathbf{x}) \in \mathbb{R}^n$.
If T is a linear transformation, then each component of the image vector $\mathbf{y}$ must be a linear function of the components of $\mathbf{x}$.
More specifically, if we write

$$T(\mathbf{x}) = \mathbf{y} = [y_1(x_1, \ldots, x_m), \ldots, y_n(x_1, \ldots, x_m)]$$

then each component function $y_i(x_1, \ldots, x_m)$ must be a linear function of the variables $x_1, \ldots, x_m$ **without constant terms**; i.e., each component function y_i a function of the form

$$y_i(x_1, \ldots, x_m) = c_{i,1}x_1 + c_{i,2}x_2 + \cdots + c_{i,m}x_m$$

Checking if a function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a Linear Transformation

Example 1:
Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3 : (s,t) \to (t, s, 1+t+s)$ is **not** a linear transformation. (Note the constant term 1 in the linear function corresponding to the third component of the image.)

Let $\mathbf{v} = (s,t)$. Then

$$T(\mathbf{v}) = T(s,t) = (t, s, 1+t+s)$$

and

$$\begin{aligned} T(r\mathbf{v}) &= T(rs, rt) = (rt, rs, 1+rs+rt) \\ rT(\mathbf{v}) &= r(t, s, 1+t+s) = (rt, rs, r+rt+rs) \end{aligned}$$

Since $T(r\mathbf{v}) \neq rT(\mathbf{v})$, **T** does not preserve scalar multiplication: hence T is not a linear transformation.

Checking if a function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a Linear Transformation, Cont'd

Example 2: Let $\mathbf{A}$ be an $n \times m$ matrix.
To any vector in $\mathbb{R}^m$, we can associate an $m \times 1$ column vector $\mathbf{x}$, and via multiplication from the left by $\mathbf{A}$, a $n \times 1$ column vector

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m \end{bmatrix} \in \mathbb{R}^n$$

Define $T_A : \mathbb{R}^m \to \mathbb{R}^n$ by

$$T_A(\mathbf{x}) = \mathbf{Ax}$$

Then

$$\begin{aligned} T_{\mathbf{A}}(\lambda \mathbf{x}) &= \mathbf{A}(\lambda \mathbf{x}) = \lambda \mathbf{Ax} = \lambda T_{\mathbf{A}}(\mathbf{x}) \\ T_{\mathbf{A}}(\mathbf{x}_1 + \mathbf{x}_2) &= \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}x_1 + \mathbf{Ax}_2 = T_{\mathbf{A}}(\mathbf{x}_1) + T_{\mathbf{A}}(\mathbf{x}_2) \end{aligned}$$

and so $T_{\mathbf{A}}$ is a linear transformation.