Lecture 19: Subspaces Attached to Linear Transformations

Math 3013
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Agenda

- Linear Transformations and Matrices
- The Range of a Linear Transformation

Linear Transformations

Definition

A **linear transformation** is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

(i) $\lambda \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^m \quad \Rightarrow \quad T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$

(ii) $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^m \quad \Rightarrow \quad T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$

Linear Transformations and Matrices

Every $n \times m$ matrix $\mathbf{A}$ defines a linear transformation $T_{\mathbf{A}} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ via

$$\mathbb{R}^m \ni \mathbf{x} \mapsto T_{\mathbf{A}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x} \in \mathbb{R}^n$$

The converse is also true:

Theorem

For every linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, there is a unique matrix $n \times m$ matrix $\mathbf{A}_T$, such that

$$T(\mathbf{x}) = \mathbf{A}_T \mathbf{x}$$

Showing this, however, is going to require a couple short digressions.

Digression: A Matrix Multiplication Identity

Lemma

Suppose $\mathbf{A}$ *is an* $m \times n$ *matrix and* $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ *is an* $m \times 1$ *column vector.*
Write $\mathbf{Col}_j(\mathbf{A})$ *for the* j^{th} *column of* $\mathbf{A}$*. Then*

$$\mathbf{Ax} = x_1\mathbf{Col}_1(\mathbf{A}) + x_2\mathbf{Col}_2(\mathbf{A}) + \cdots + x_m\mathbf{Col}_m(\mathbf{A})$$

Note that this theorem tells us that $\mathbf{Ax}$ is always a vector in $ColSp\,(\mathbf{A})$, the column space of $\mathbf{A}$.

Proof of the Lemma

$$
\begin{aligned}
\mathbf{Ax} &= \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \\
&= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix} \\
&= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix} \\
&= x_1 \mathbf{Col}_1(\mathbf{A}) + x_2 \mathbf{Col}_2(\mathbf{A}) + \cdots + x_m \mathbf{Col}_m(\mathbf{A})
\end{aligned}
$$

□

Digression 2: The Standard Basis for $\mathbb{R}^m$

Recall the standard basis vectors for $\mathbb{R}^m$.

$$\begin{aligned} \mathbf{e}_1 &= [1, 0, 0, \ldots, 0, 0] \\ \mathbf{e}_2 &= [0, 1, 0, \ldots, 0, 0] \\ &\vdots \\ \mathbf{e}_{m-1} &= [0, 0, 0, \ldots, 1, 0] \\ \mathbf{e}_m &= [0, 0, 0, \ldots, 0, 1] \end{aligned}$$

Let $\mathcal{E} = \{\mathbf{e}_1, \ldots \mathbf{e}_m\}$. Note that

(i) $\mathbb{R}^m = span\,(\mathbf{e}_1, \ldots, \mathbf{e}_m)$

(ii) the vectors in $\mathcal{E}$ are linearly independent

and so $\mathcal{E}$ is a basis for $\mathbb{R}^m$.

We call $\mathcal{E}$ the **standard basis** for $\mathbb{R}^m$. Just as with any basis for a subspace, we can expand a vector $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{R}^m$, as a linear combination of the **standard basis vectors** $\mathbf{e}_1, \ldots, \mathbf{e}_m$

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m$$

Digression 2: The Standard Basis for $\mathbb{R}^m$, Cont'd

What's special about the standard basis is that the coordinate vector of $\mathbf{x}$ with respect to the standard basis $\mathcal{E}$, is the same list of numbers as $\mathbf{x}$ itself :

$$\begin{aligned} \mathbb{R}^m \ni \mathbf{x} &= [x_1, \ldots, x_m] \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m \end{aligned}$$

$$\Longrightarrow \qquad \mathbf{x}_{\mathcal{E}} = [x_1, x_2, \ldots, x_m]$$

Linear Transformations and Matrices - Theorem

Theorem

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ *be a linear transformation and let* $\mathbf{A}_T$ *be the* $n \times m$ *matrix formed by using the vectors* $T(\mathbf{e}_i) \in \mathbb{R}^n$ *(*$i = 1, \ldots, m$*) as columns*

$$\mathbf{A}_T = \left[\begin{array}{cccc} \uparrow & \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \downarrow & \cdots & \downarrow \end{array} \right]$$

Then

$$T(\mathbf{x}) = \mathbf{A}_T \mathbf{x}$$

where, on the right, we interpret $\mathbf{x} \in \mathbb{R}^m$ *as an* $m \times 1$ *column vector.*

Proof of Theorem

We want to show that if

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

then

$$T(\mathbf{x}) = \mathbf{A}_T \mathbf{x}$$

We have

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m) && \text{(expansion of } \mathbf{x} \text{ w.r.t. } \mathcal{E}) \\ &= T(x_1\mathbf{e}_1) + \cdots + T(x_m\mathbf{e}_m) && (T \text{ preserves vector addition}) \\ &= x_1 T(\mathbf{e}_1) + \cdots + x_m T(\mathbf{e}_m) && (T \text{ preserves scalar mult.}) \\ &= x_1 \mathbf{Col}_1(\mathbf{A}_T) + \cdots + x_m \mathbf{Col}_m(\mathbf{A}_T) && \text{(definition of } \mathbf{A}_T) \\ &= \mathbf{A}_T \mathbf{x} && \text{(by our matrix multiplication identity)} \end{aligned}$$

Example

Consider

$$T : \mathbb{R}^3 \to \mathbb{R}^2 : [x_1, x_2, x_3] \longmapsto [x_1 - x_2, x_2 + x_3]$$

Let's verify that this is a linear transformation

$$\begin{aligned}
T(\lambda \mathbf{x}) &= T(\lambda [x_1, x_2, x_3]) \\
&= T([\lambda x_1, \lambda x_2, \lambda x_3]) \\
&= [\lambda x_1 - \lambda x_2, \lambda x_2 + \lambda x_3] \\
&= \lambda [x_1 - x_2, x_2 + x_3] \\
&= \lambda T([x_1, x_2, x_3]) \\
&= \lambda T(\mathbf{x})
\end{aligned}$$

and so T is compatible with scalar multiplication.

Example: $T : \mathbb{R}^3 \to \mathbb{R}^2 : [x_1, x_2, x_3] \longmapsto [x_1 - x_2, x_2 + x_3]$, Cont'd

Now let's examine how T behaves w.r.t. vector addition:

$$
\begin{aligned}
T(\mathbf{x} + \mathbf{y}) &= T([x_1, x_2, x_3] + [y_1, y_2, y_3]) \\
&= T([x_1 + y_1 \;,\; x_2 + y_2 \;,\; x_3 + y_3]) \\
&= [(x_1 + y_1) - (x_2 + y_2) \;,\; (x_2 + y_2) + (x_3 + y_3)] \\
&= [(x_1 - x_2) + (y_1 - y_1) \;,\; (x_2 + x_3) + (y_2 + y_3)] \\
&= [x_1 - x_2 \;,\; x_2 + x_3] + [y_1 - y_2 \;,\; y_2 + y_3] \\
&= T(\mathbf{x}) + T(\mathbf{y})
\end{aligned}
$$

and so T is also compatible with vector addition.

Being compatible with both scalar multiplication and vector addition, T is thus a linear transformation. □

Example: $T : \mathbb{R}^3 \to \mathbb{R}^2 : [x_1, x_2, x_3] \longmapsto [x_1 - x_2, x_2 + x_3]$, Cont'd

Constructing the corresponding matrix $\mathbf{A}_T$:
We have

$$\begin{aligned} T(\mathbf{e}_1) &= T([1,0,0]) = [1-0, 0+0] = [1,0] \\ T(\mathbf{e}_2) &= T([0,1,0]) = [0-1, 1+0] = [-1,1] \\ T(\mathbf{e}_3) &= T([0,0,1]) = [0-0, 0+1] = [0,1] \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{A}_T &= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

Example: $T : \mathbb{R}^3 \to \mathbb{R}^2 : [x_1, x_2, x_3] \longmapsto [x_1 - x_2, x_2 + x_3]$, Cont'd

And so

$$\mathbf{A}_T \mathbf{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + 0 \\ 0 + x_2 + x_3 \end{bmatrix}$$

which agrees with the value of $T(\mathbf{x})$ component-by-component.

Upshot: Linear Transformations and Matrices

- Just as **subspaces** of a vector space $\mathbb{R}^m$ are a special class of subsets of $\mathbb{R}^n$; characterized by their behavior w.r.t. scalar multiplication and vector addition,
linear transformations are a special class of functions characterized by their behavior w.r.t. scalar multiplication and vector addition,

- Because of the one-to-one correspondence

$$\text{Linear Transf.}\, T : \mathbb{R}^m \to \mathbb{R}^n \longleftrightarrow n \times m \text{ matrix } \mathbf{A}$$

questions about linear transformations can be addressed within the calculational setting of $n \times m$ matrices.

Subspaces Attached to a Linear Transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

Recall that, given an $n \times m$ matrix $\mathbf{A}$, there are three associated subspaces

$$\begin{aligned} RowSp(\mathbf{A}) &= \text{span of the row vectors of } \mathbf{A} \\ ColSp(\mathbf{A}) &= \text{span of the column vectors of } \mathbf{A} \\ Null(\mathbf{A}) &= \text{solution set of } \mathbf{Ax} = \mathbf{0} \end{aligned}$$

Given the close connection between matrices and linear transformations, one should expect that there are also subspaces attached to a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$.

The Range of a Linear Transformation

Definition

If $f : A \to B$ is a function between two sets, the subset of B defined by

$$Image\,(f) = \{b \in B \mid b = f\,(a) \text{ for some } a \in A\}$$

is called the **image** of the function f (in B).

In the special case of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, the image of T is called the **range** of T:

$$Range\,(T) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T\,(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^m\}$$

Lemma

If $T : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear transformation, then* $Range\,(T)$ *is a subspace of* $\mathbb{R}^n$.

Proof. $Range\,(T)$ is already defined as a subset of $\mathbb{R}^n$.

We have to show that $Range\,(T)$ is closed under both scalar multiplication and vector addition.

Range(T) : Closure Under Scalar Multiplication

Let $\mathbf{y} \in Range\,(T)$ and let $\lambda \in \mathbb{R}$.
We want to show that $\lambda\mathbf{y}$ is always in $Range\,(T)$.
By definition, $\mathbf{y} = T\,(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^m$.

Now consider

$$T\,(\lambda\mathbf{x}) = \lambda T\,(\mathbf{x}) = \lambda\mathbf{y}$$

This shows that $\lambda\mathbf{y}$ is in $Range\,(T)$; is the image of the point $\lambda\mathbf{x} \in \mathbb{R}^m$.
Thus, $Range\,(T)$ is closed under scalar multiplication.

Range(T) : Closure Under Vector Addition

Let $\mathbf{y}_1, \mathbf{y}_2 \in Range\,(T)$.
Then, by definition, there exists vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$ such that

$$\mathbf{y}_1 = T\,(\mathbf{x}_1) \qquad \text{and} \qquad \mathbf{y}_2 = T\,(\mathbf{x}_2)$$

Now consider

$$\mathbf{T}\,(\mathbf{x}_1 + \mathbf{x}_2) = T\,(\mathbf{x}_1) + T\,(\mathbf{x}_2) = \mathbf{y}_1 + \mathbf{y}_2$$

This shows that $\mathbf{y}_1 + \mathbf{y}_2 \in Range\,(T)$ since it is the image of $\mathbf{x}_1 + \mathbf{x}_2$ by T.
Thus, *Range*$\,(T)$ is also closed under vector addition.

Conclusion of Proof

Since $Range(T)$ is a subset of $\mathbb{R}^n$ that is

(i) closed under scalar multiplication and

(ii) closed under vector addition

$Range(T)$ is a subspace of $\mathbb{R}^n$.

$\square$