Lecture 20: Linear Transformations, Cont'd

Math 3013
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Agenda

1. Linear Transformations and Matrices
2. The Range of a Linear Transformation
3. The Kernel of a Linear Transformation
4. The Dimensions of $Range(T)$ and $Ker(T)$
5. Composition of Linear Transformations

Review: Linear Transformations and Matrices

A **linear transformation** is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ between two vector spaces such that

(i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\lambda \in \mathbb{R}$, and all $\mathbf{x} \in \mathbb{R}^m$

(ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

Given an $n \times m$ matrix $\mathbf{A}$, one can construct a corresponding linear transformation $T_{\mathbf{A}} : \mathbb{R}^m \to \mathbb{R}^n$ by setting

$$T_{\mathbf{A}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}$$

Conversely, given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ one can construct a matrix $\mathbf{A}_T$ such that

$$T(\mathbf{x}) = \mathbf{A}_T \mathbf{x}$$

by setting

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \cdots & \downarrow \end{bmatrix}$$

($\mathbf{e}_1, \ldots, \mathbf{e}_m$ being the standard basis vectors for the domain $\mathbb{R}^m$).

Subspaces Attached to a Linear Transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

Recall that, given an $n \times m$ matrix $\mathbf{A}$, there are three associated subspaces

$$\begin{aligned} RowSp(\mathbf{A}) &= \text{span of the row vectors of } \mathbf{A} \\ ColSp(\mathbf{A}) &= \text{span of the column vectors of } \mathbf{A} \\ Null(\mathbf{A}) &= \text{solution set of } \mathbf{Ax} = \mathbf{0} \end{aligned}$$

Given the close connection between matrices and linear transformations, one should expect that there are also subspaces attached to a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$.

The Range of a Linear Transformation

Definition

If $f : A \to B$ is a function between two sets, the subset of B defined by

$$Image\,(f) \equiv \{b \in B \mid b = f\,(a) \text{ for some } a \in A\}$$

is called the **image** of the function f (in B).

In the special case of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, the image of T is called the **range** of T:

$$Range\,(T) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T\,(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^m\}$$

Lemma

If $T : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear transformation, then* $Range\,(T)$ *is a subspace of* $\mathbb{R}^n$.

Proof. $Range\,(T)$ is already defined as a subset of $\mathbb{R}^n$.

We have to show that $Range\,(T)$ is closed under both scalar multiplication and vector addition:

$$\begin{aligned} \lambda \in \mathbb{R}, \mathbf{y} \in Range(T) \quad &\Longrightarrow \quad \lambda\mathbf{y} \in Range(T) \\ \mathbf{y}_1, \mathbf{y}_2 \in Range(T) \quad &\Longrightarrow \quad \mathbf{y}_1 + \mathbf{y}_2 \in Range(T) \end{aligned}$$

$Range(T)$: Closure Under Scalar Multiplication

Let $\mathbf{y} \in Range\,(T)$ and let $\lambda \in \mathbb{R}$.
We want to show that $\lambda\mathbf{y}$ is always in $Range\,(T)$.
Since $\mathbf{y} \in Range(T)$, there exists an $\mathbf{x} \in \mathbb{R}^m$ such that

$$\mathbf{y} = T\,(\mathbf{x}) \qquad (*)$$

Now multiply both sides of (*) by λ

$$\begin{aligned} \lambda\mathbf{y} &= \lambda T\,(\mathbf{x}) \\ &= T\,(\lambda\mathbf{x}) \qquad \text{(because } T \text{ is L.T.)} \end{aligned}$$

This shows that

$$\lambda \in \mathbb{R}, \mathbf{y} \in Range(T) \quad \Longrightarrow \quad \lambda\mathbf{y} \in Range(T)$$

i.e., $Range(T)$ is closed under scalar multiplication. $\square$

Range(T) : Closure Under Vector Addition

Let $\mathbf{y}_1, \mathbf{y}_2 \in Range\,(T)$.
Then there must be vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$ such that

$$\begin{aligned} \mathbf{y}_1 &= T\,(\mathbf{x}_1) \\ \mathbf{y}_2 &= T\,(\mathbf{x}_2) \end{aligned}$$

Adding these two equations yields

$$\begin{aligned} \mathbf{y}_1 + \mathbf{y}_2 &= T\,(\mathbf{x}_1) + T\,(\mathbf{x}_2) \\ &= T\,(\mathbf{x}_1 + \mathbf{x}_2) \quad \text{(because } T \text{ is a L.T.)} \end{aligned}$$

This shows

$$\mathbf{y}_1, \mathbf{y}_2 \in Range(T) \quad \Longrightarrow \quad \mathbf{y}_1 + \mathbf{y}_2 \in Range(T)$$

Thus, *Range*$\,(T)$ is also closed under vector addition. □

Conclusion of Proof

Since $Range(T)$ is a subset of $\mathbb{R}^n$ that is

(i) closed under scalar multiplication and

(ii) closed under vector addition

$Range(T)$ is a subspace of $\mathbb{R}^n$.

$\square$

How to Find a Basis for *Range* (T)

Lemma

Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear transformation and let*

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \cdots & \downarrow \end{bmatrix}$$

be the associated $n \times m$ *matrix.*
Then

$$Range\,(T) = ColSp\,(\mathbf{A}_T)$$

Proof of Lemma:

Idea of Proof: If two sets are equal, each has to be a subset of the other.

$$Range\,(T) = ColSp\,(\mathbf{A}_T) \quad \Longleftrightarrow \quad \begin{cases} Range\,(T) \subset ColSp\,(\mathbf{A}_T) \\ ColSp\,(\mathbf{A}_T) \subset Range\,(T) \end{cases}$$

Proof, Part 1: $Range\left(T\right) \subset ColSp\left(\mathbf{A}_T\right)$

Let $\mathbf{y} \in Range\left(T\right)$.
Then $\mathbf{y} = T\left(\mathbf{x}\right)$ for some $\mathbf{x} \in \mathbb{R}^m$.
Expanding $\mathbf{x}$ with respect to the standard basis of $\mathbb{R}^m$, we see

$$\begin{aligned}
\mathbf{y} &= T\left(\mathbf{x}\right) \\
&= T\left(x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m\right) \\
&= T\left(x_1\mathbf{e}_1\right) + \cdots + T\left(x_m\mathbf{e}_m\right) \\
&= x_1 T\left(\mathbf{e}_1\right) + \cdots + x_m T\left(\mathbf{e}_m\right) \\
&= x_1\mathbf{Col}_1\left(\mathbf{A}_T\right) + \cdots + x_m\mathbf{Col}_m\left(\mathbf{A}_T\right) \\
&\in ColSp\left(\mathbf{A}_T\right)
\end{aligned}$$

So every $\mathbf{y} \in Range\left(T\right)$ is also in the column space of $\mathbf{A}_T$.

Proof, Part 2: $ColSp(\mathbf{A}_T) \subset Range(T)$

Now consider a point $\mathbf{w} \in ColSp(\mathbf{A}_T)$. We have

$$\begin{aligned} \mathbf{w} &\in span\left(\mathbf{Col}_1(\mathbf{A}_T), \ldots, \mathbf{Col}_m(\mathbf{A}_T)\right) \\ &\Rightarrow \quad \mathbf{w} = c_1\mathbf{Col}_1(\mathbf{A}_T) + \cdots + c_m\mathbf{Col}_m(\mathbf{A}_T) \end{aligned}$$

for some coefficients $c_1, \ldots, c_m$.
But then

$$\begin{aligned} \mathbf{w} &= c_1 T(\mathbf{e}_1) + \cdots + c_m T(\mathbf{e}_m) \\ &= T(c_1\mathbf{e}_1) + \cdots + T(c_m\mathbf{e}_m) \\ &= T(c_1\mathbf{e}_1 + \cdots + c_m\mathbf{e}_m) \\ &\in Range(T) \end{aligned}$$

Proof of Lemma, Cont'd

Thus, we have now shown that

- every element of $Range\,(T)$ is an element of $ColSp\,(\mathbf{A}_T)$ and
- every element of $ColSp\,(\mathbf{A}_T)$ is an element of $Range\,(T)$.

And so the two sets coincide:

$$Range\,(T) = ColSp\,(\mathbf{A}_T)$$

$\square$

Corollary

A basis for $Range\,(T)$ can be found by finding a basis for the column space of $\mathbf{A}_T$.

Example

Consider $T : \mathbb{R}^2 \to \mathbb{R}^3 : [x_1, x_2] \longmapsto [x_1 + x_2, x_1 - x_2, x_1]$. Find a basis for $Range\,(T)$.

Let's first compute $\mathbf{A}_T$.

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Now we can apply the preceding corollary and our method of finding a basis for the column space of a matrix.

Example: finding basis for *Range*(T), Cont'd

$\mathbf{A}_T$ row reduces to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R.R.E.F.(\mathbf{A}_T)$$

Since both columns of the R.R.E.F. contain pivots, the corresponding columns of $\mathbf{A}_T$ will provide a basis for $ColSp(\mathbf{A}_T) = Range(T)$.
Thus,

$$\begin{aligned} \text{basis for } Range(T) &= \text{basis for } ColSp(\mathbf{A}_T) \\ &= \{[1,1,1], [1,-1,0]\} \end{aligned}$$

The Kernel of a Linear Transformation

Definition

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The **kernel** $\ker(T)$ of a linear transformation is the subset of the domain $\mathbb{R}^m$ given by

$$\ker(T) = \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}\}$$

Lemma

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and let $\mathbf{A}_T$ be the associated matrix.
Then

$$\ker(T) = Null(\mathbf{A}_T)$$

Proof of Lemma

Suppose $\mathbf{x} \in \ker(T)$, then $T(\mathbf{x}) = \mathbf{0}$. But if

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \cdots & \downarrow \end{bmatrix}$$

we have

$$T(\mathbf{x}) = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{A}_T\mathbf{x} = \mathbf{0}$$

and so every $\mathbf{x} \in \ker(T)$ is also in $Null(\mathbf{A}_T)$ and every $\mathbf{x} \in Null(\mathbf{A}_T)$ is also in $\ker(T)$.
Thus, the two sets coincide.
In fact, since $Null(\mathbf{A}_T)$ is known to be a subspace of $\mathbb{R}^m$, it follows that $\ker(T)$ is a subspace of $\mathbb{R}^m$.

Example: Computing a basis for *ker*(T)

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$:
$T([x_1, x_2, x_3]) = [x_1 - x_2, x_2 - x_3]$.
Find a basis for $\ker(T)$.

We begin by calculating the associated matrix $\mathbf{A}_T$

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

We need to calculate a basis for $Null(\mathbf{A}_T)$, or equivalently, a basis for the solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$.

Example, Cont'd

Row reducing $\mathbf{A}_T$ to its R.R.E.F.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{array}{r} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = x_3 \\ x_2 = x_3 \end{array}\right.$$

$$\Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The constant vector being multiplied by the free parameter x_3 is our basis vector.
Thus,

$$\text{basis for } \ker(T) = \{[1, 1, 1]\}$$

The Dimensions of *Range* (T) and *Ker* (T)

Recall

Theorem

If $\mathbf{A}$ *is an* $n \times m$ *matrix*

$$\#\mathit{columns\ of}\ \mathbf{A} = \mathit{Rank}\,(\mathbf{A}) + \mathit{Nullity}\,(\mathbf{A})$$

where

$$\begin{aligned}\mathit{Rank}\,(\mathbf{A}) &= \dim\left(\mathit{RowSp}\,(\mathbf{A})\right) = \dim\left(\mathit{ColSp}\,(\mathbf{A})\right) \\ &= \#\,(\text{columns with pivots in any R.E.F. of } \mathbf{A}\ )\end{aligned}$$

$$\begin{aligned}\mathit{Nullity}\,(\mathbf{A}) &= \dim\left(\mathit{NullSp}\,(\mathbf{A})\right) \\ &= \dim\left(\text{solution set of } \mathbf{Ax} = \mathbf{0}\right) \\ &= \#\,(\text{free parameters in general solution}) \\ &= \#\,(\text{columns without pivots in any R.E.F. of } \mathbf{A})\end{aligned}$$

Dimension Formula for Subspaces attached to a Linear Transformation

Theorem

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then

$$m = \dim\left(Range(T)\right) + \dim\left(Kernel(T)\right)$$

Proof Let $\mathbf{A}_T$ be the $n \times m$ matrix attached to $T : \mathbb{R}^m \to \mathbb{R}^n$.

$$\begin{aligned}
m &= \#\left(\text{columns of } \mathbf{A}_T\right) \\
&= \#\left(\text{columns of any R.E.F. of } \mathbf{A}_T \text{ with pivots}\right) \\
&\quad +\#\left(\text{columns any R.E.F. of } \mathbf{A}_T \text{ without pivots}\right) \\
&= \dim\left(ColSp\left(\mathbf{A}_T\right)\right) + \dim\left(NullSp\left(\mathbf{A}_T\right)\right) \\
&= \dim\left(Range\left(T\right)\right) + \dim\left(Kernel\left(T\right)\right)
\end{aligned}$$

$\square$

Composition of Linear Transformations

Theorem

If $T:\mathbb{R}^m \to \mathbb{R}^n$ and $S:\mathbb{R}^n \to \mathbb{R}^p$ are linear transformations, then the composed function

$$S \circ T : \mathbb{R}^m \to \mathbb{R}^p : S \circ T(\mathbf{x}) = S(T(\mathbf{x}))$$

is also a linear transformation. Moreover, the $p \times m$ matrix $\mathbf{A}_{S\circ T}$ attached to the linear transformation $S \circ T$ can be computed as

$$\mathbf{A}_{S\circ T} = \mathbf{A}_S \mathbf{A}_T$$

Example

Consider the linear transformations

$$\begin{aligned} T &: \mathbb{R}^2 \to \mathbb{R}^3 : T([x_1, x_2]) = [x_2, x_1, x_1 + x_2] \\ S &: \mathbb{R}^3 \to \mathbb{R}^2 : S([y_1, y_2, y_3]) = [y_1 + y_2, y_3] \end{aligned}$$

We have

$$S \circ T([x_1, x_2]) = S([x_2, x_1, x_1 + x_2]) = [x_2 + x_1, x_1 + x_2]$$

Thus,

$$\mathbf{A}_{S \circ T} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

On the other hand,

$$\mathbf{A}_T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad , \quad \mathbf{A}_S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{A}_S \mathbf{A}_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \mathbf{A}_{S \circ T}$$