Lecture 21: Vector Space Isomorphisms

Math 3013
Oklahoma State University

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Agenda

1. Range and Kernel of Linear Transformations
2. Invertibility of Linear Transformations
3. Vector Space Isomorphisms
4. Examples

Review: Linear Transformations and Matrices

A **linear transformation** is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ between two vector spaces such that

(i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\lambda \in \mathbb{R}$, and all $\mathbf{x} \in \mathbb{R}^m$

(ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

Linear Transformations $\Longleftrightarrow$ Matrices :

- $n \times m$ matrix $\mathbf{A} \Longrightarrow$ linear transformation $T_{\mathbf{A}} : \mathbb{R}^m \to \mathbb{R}^n$:

$$T_{\mathbf{A}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}$$

- Linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n \Longrightarrow n \times m$ matrix $\mathbf{A}_T$

$$\mathbf{A}_T \equiv \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \cdots & \downarrow \end{bmatrix}$$

Range and Kernel of a Linear Transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

$$\begin{aligned} Range(T) &\equiv \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\} \\ &\approx ColSp(\mathbf{A}_T) \end{aligned}$$

$$\begin{aligned} Ker(T) &\equiv \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = 0\} \\ &\approx NullSp(\mathbf{A}_T) \\ &\equiv \text{ solution set of } \mathbf{A}_T\mathbf{x} = \mathbf{0} \end{aligned}$$

The Dimensions of *Range* (T) and *Ker* (T)

Recall

Theorem

If $\mathbf{A}$ *is an* $n \times m$ *matrix*

$$\#\mathit{columns\ of}\ \mathbf{A} = \mathit{Rank}\,(\mathbf{A}) + \mathit{Nullity}\,(\mathbf{A})$$

where

$$\begin{aligned}\mathit{Rank}\,(\mathbf{A}) &= \dim\left(\mathit{RowSp}\,(\mathbf{A})\right) = \dim\left(\mathit{ColSp}\,(\mathbf{A})\right)\\ &= \#\,(\text{columns with pivots in any R.E.F. of } \mathbf{A}\ )\end{aligned}$$

and

$$\begin{aligned}\mathit{Nullity}\,(\mathbf{A}) &= \dim\left(\mathit{NullSp}\,(\mathbf{A})\right)\\ &= \dim\left(\text{solution set of } \mathbf{Ax} = \mathbf{0}\right)\\ &= \#\,(\text{free parameters in general solution})\\ &= \#\,(\text{columns without pivots in any R.E.F. of } \mathbf{A})\end{aligned}$$

Dimension Formula for Subspaces attached to a Linear Transformation

Theorem

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then

$$m = \dim\left(Range(T)\right) + \dim\left(Kernel(T)\right)$$

Proof Let $\mathbf{A}_T$ be the $n \times m$ matrix attached to $T : \mathbb{R}^m \to \mathbb{R}^n$.

$$\begin{aligned}
m &= \#\left(\text{columns of } \mathbf{A}_T\right) \\
&= \#\left(\text{columns of any R.E.F. of } \mathbf{A}_T \text{ with pivots}\right) \\
&\quad + \#\left(\text{columns any R.E.F. of } \mathbf{A}_T \text{ without pivots}\right) \\
&= \dim\left(ColSp\left(\mathbf{A}_T\right)\right) + \dim\left(NullSp\left(\mathbf{A}_T\right)\right) \\
&= \dim\left(Range\left(T\right)\right) + \dim\left(Kernel\left(T\right)\right)
\end{aligned}$$

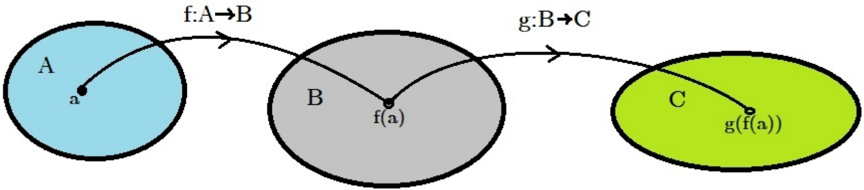
Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two functions such that

$$Image(f) \subset Domain(g)$$

Then the **composition** of f and g is the function $f \circ g : A \to C$ defined by

$$(f \circ g)(a) = g(f(a)) \quad \forall a \in A$$



Composition of Linear Transformations

Theorem

If $T : \mathbb{R}^m \to \mathbb{R}^n$ *and* $S : \mathbb{R}^n \to \mathbb{R}^p$ *are linear transformations, then the composed function*

$$S \circ T : \mathbb{R}^m \to \mathbb{R}^p : S \circ T(\mathbf{x}) = S(T(\mathbf{x}))$$

is also a linear transformation. Moreover, the $p \times m$ *matrix* $\mathbf{A}_{S \circ T}$ *attached to the linear transformation* $S \circ T$ *can be computed as*

$$\mathbf{A}_{S \circ T} = \mathbf{A}_S \mathbf{A}_T$$

Example

Consider the linear transformations

$$\begin{aligned} T &: \mathbb{R}^2 \to \mathbb{R}^3 : T([x_1, x_2]) = [x_2, x_1, x_1 + x_2] \\ S &: \mathbb{R}^3 \to \mathbb{R}^2 : S([y_1, y_2, y_3]) = [y_1 + y_2, y_3] \end{aligned}$$

We have

$$S \circ T([x_1, x_2]) = S([x_2, x_1, x_1 + x_2]) = [x_2 + x_1, x_1 + x_2]$$

Thus,

$$\mathbf{A}_{S \circ T} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

On the other hand,

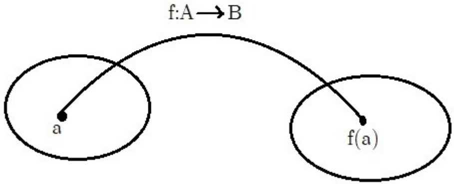
$$\mathbf{A}_T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad , \quad \mathbf{A}_S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{A}_S \mathbf{A}_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \mathbf{A}_{S \circ T}$$

Invertibility of Functions

Let $f : A \to B$ be a function between two sets A and B.



Recall

$$Image(f) \equiv \{b \in B \mid b = f(a) \text{ for some } a \in A\}$$

Definition

A function $f : A \to B$ is called **surjective** if $Image(f) = B$

In other words, $f : A \to B$ is surjective if every point b in the codomain B is reachable from A via f.

Invertibility of Functions, Cont'd

Definition

A function $f : A \to B$ is called **injective** if

$$f(a) = f(a') \qquad \Longrightarrow \qquad a = a'$$

In other words, $f : A \to B$ is injective if distinct points in the domain A get mapped to distinct points in the codomain B.

Surjectivity and Injectivity: Examples

Consider

$$f : \mathbb{R} \to \mathbb{R} : f(x) = x^2$$

This function is neither surjective nor injective.
f is not surjective because a negative number (regarded as an element of the codomain) can not be reached by applying f to any element x in the domain of f.
f is not injective because, for example,

$$f(1) = 1 = f(-1) \quad \text{but} \quad 1 \neq -1$$

However, if we change just the domain and codomain of f we can readily produce a function that is bijective. For example, let

$$\mathbb{R}_+ \equiv \{x \in \mathbb{R} \mid x > 0\}$$

Then

$$\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+ : x \mapsto x^2$$

is both surjective and injective.

Bijective Functions and Invertibility

Definition

A function $f : A \to B$ is called **bijective** if f is both surjective and injective

Definition

A function $f : A \to B$ is called **invertible** if there is a function $f^{-1} : B \to A$ such that

$$\begin{aligned} \left(f \circ f^{-1}\right)(b) &= b \quad \forall b \in B \\ \left(f^{-1} \circ f\right)(a) &= a \quad \forall a \in A \end{aligned}$$

Theorem

A function is invertible if and only if it is bijective.

Invertibility of Linear Transformations

Remember that a **linear transformation** is a special kind of function: it is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\begin{aligned} \lambda \in \mathbb{R}\,,\ \mathbf{x} \in \mathbb{R}^m \quad &\Longrightarrow \quad T(\lambda \mathbf{x}) = \lambda T(\mathbf{x}) \\ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m \quad &\Longrightarrow \quad T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2) \end{aligned}$$

Lemma

A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is surjective if and only if $Range(T) = \mathbb{R}^n$.

(This follows from the fact that *Range*(T) is the just linear algebra terminology for *image*(T) as a function between two vector spaces.)

Lemma

A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ *is injective if and only if* $ker(T) = \{\mathbf{0}\}$.

Proof. We first note that for any linear transformation $T(\mathbf{0}) = \mathbf{0}$. To see this choose any $\mathbf{x} \in \mathbb{R}^m$. Then

$$\begin{aligned} T(\mathbf{0}) &= T(\mathbf{x} - \mathbf{x}) \\ &= T(\mathbf{x}) - T(\mathbf{x}) \\ &= \mathbf{0} \end{aligned}$$

Proof: $\Longrightarrow$

Suppose T is injective. By definition,

$$ker(T) = \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}\}$$

Suppose we had a vector $\mathbf{x}$ besides $\mathbf{0}$, such that $T(\mathbf{x}) = \mathbf{0}$. Then we would have

$$T(\mathbf{x}) = \mathbf{0} = T(\mathbf{0}) \qquad \text{but} \quad \mathbf{x} \neq \mathbf{0}$$

which would contradict T being injective.
Therefore, $\mathbf{0}$ must be the only vector whose value is $\mathbf{0}$. Hence, $ker(T) = \{\mathbf{0}\}$. $\square$

Proof: $\Longleftarrow$

Suppose $ker(T) = \{\mathbf{0}\}$. We want to show that T is injective. Suppose

$$T(\mathbf{x}_1) = T(\mathbf{x}_2)$$

Then

$$\begin{aligned} \mathbf{0} &= T(\mathbf{x}_1) - T(\mathbf{x}_2) \\ &= T(\mathbf{x}_1 - \mathbf{x}_2) \end{aligned}$$

But if $ker(T) = \{\mathbf{0}\}$, then we must have

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$$

because no other vector can attain the value $\mathbf{0}$. Thus, when $ker(T) = \{\mathbf{0}\}$,

$$T(\mathbf{x}_1) = T(\mathbf{x}_2) \qquad \Longrightarrow \qquad \mathbf{x}_1 = \mathbf{x}_2$$

and so T is injective. $\square$

Vector Space Isomorphisms

Recall that a function $f : A \to B$ is invertible if it is both surjective and injective.
For linear transformations then (using the appropriate nomenclature for linear transformations)

Theorem

A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is invertible if and only if

(i) $Range(T) = \mathbb{R}^n$ *(T is surjective)*

(ii) $Ker(T) = \{\mathbf{0}\}$ *(T is injective)*

Definition

An invertible linear transformation is called a **vector space isomorphism**