Lecture 22: Vector Space Isomorphisms

Math 3013
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Agenda

1. Linear Transformations and Their Associated Subspaces
2. Invertibility of Linear Transformations
3. Vector Space Isomorphisms
4. General Vector Spaces

Review: Linear Transformations and Matrices

A **linear transformation** is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ between two vector spaces such that

(i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\lambda \in \mathbb{R}$, and all $\mathbf{x} \in \mathbb{R}^m$

(ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

Linear Transformations $\Longleftrightarrow$ Matrices :

- $n \times m$ matrix $\mathbf{A} \Longrightarrow$ linear transformation $T_{\mathbf{A}} : \mathbb{R}^m \to \mathbb{R}^n$:

$$T_{\mathbf{A}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}$$

- Linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n \Longrightarrow n \times m$ matrix $\mathbf{A}_T$

$$\mathbf{A}_T \equiv \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \cdots & \downarrow \end{bmatrix}$$

Range and Kernel of a Linear Transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

$$
\begin{aligned}
Range\,(T) &\equiv \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T\,(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\} \\
&\approx ColSp\,(\mathbf{A}_T)
\end{aligned}
$$

$$
\begin{aligned}
Ker\,(T) &\equiv \{\mathbf{x} \in \mathbb{R}^m \mid T\,(\mathbf{x}) = 0\} \\
&\approx NullSp\,(\mathbf{A}_T) \\
&\equiv \text{ solution set of } \mathbf{A}_T\mathbf{x} = \mathbf{0}
\end{aligned}
$$

The Dimensions of *Range* (T) and *Ker* (T)

Recall

Theorem

If $\mathbf{A}$ *is an* $n \times m$ *matrix*

$$\#columns\ of\ \mathbf{A} = Rank\,(\mathbf{A}) + Nullity\,(\mathbf{A})$$

where

$$\begin{aligned} Rank\,(\mathbf{A}) &= \dim\left(RowSp\,(\mathbf{A})\right) = \dim\left(ColSp\,(\mathbf{A})\right) \\ &= \#\,(\text{columns with pivots in any R.E.F. of } \mathbf{A}\ ) \end{aligned}$$

and

$$\begin{aligned} Nullity\,(\mathbf{A}) &= \dim\left(NullSp\,(\mathbf{A})\right) \\ &= \dim\left(\text{solution set of } \mathbf{Ax} = \mathbf{0}\right) \\ &= \#\,(\text{free parameters in general solution}) \\ &= \#\,(\text{columns without pivots in any R.E.F. of } \mathbf{A}) \end{aligned}$$

Dimension Formula for Subspaces attached to a Linear Transformation

Theorem

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then

$$m = \dim\left(Range(T)\right) + \dim\left(Kernel(T)\right)$$

Proof Let $\mathbf{A}_T$ be the $n \times m$ matrix attached to $T : \mathbb{R}^m \to \mathbb{R}^n$.

$$\begin{aligned}
m &= \#\left(\text{columns of } \mathbf{A}_T\right) \\
&= \#\left(\text{columns of any R.E.F. of } \mathbf{A}_T \text{ with pivots}\right) \\
&\quad + \#\left(\text{columns any R.E.F. of } \mathbf{A}_T \text{ without pivots}\right) \\
&= \dim\left(ColSp\left(\mathbf{A}_T\right)\right) + \dim\left(NullSp\left(\mathbf{A}_T\right)\right) \\
&= \dim\left(Range\left(T\right)\right) + \dim\left(Kernel\left(T\right)\right)
\end{aligned}$$

Vector Spaces over $\mathbb{R}$

Definition

A **vector space over** $\mathbb{R}$ is a set V for which the following operations are defined

- **scalar multiplication**: for every $\lambda \in \mathbb{R}$ and $\mathbf{v} \in V$ we have a map $(\lambda, \mathbf{v}) \to \lambda\mathbf{v} \in V$.
- **vector addition:** for every pair of vectors $\mathbf{u}, \mathbf{v} \in V$ we have a map $(\mathbf{u}, \mathbf{v}) \to \mathbf{u} + \mathbf{v} \in V$

It is also necessary that these operations of vector addition and scalar multiplication together satisfy 8 axioms.

Vector Space Axioms

1. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associativity of vector addition)
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (commutativity of vector addition)
3. There exists an element $\mathbf{0}\in V$ such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in V$. (additive identity.)
4. For each $\mathbf{v}\in V$ there exists an element $-\mathbf{v}\in V$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$. (additive inverses)
5. $\lambda\,(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\lambda\mathbf{v}$ (distributivity of scalar multipliciation over vector addition).
6. $(\lambda+\mu)\,\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}$. (distributivity of scalar multiplication over addition of scalars)
7. $\lambda(\mu\mathbf{v})=(\lambda\mu)\mathbf{v}$ (scalar multiplication preserves associativity of multiplication in $\mathbb{R}$.)
8. $(1)\mathbf{v}=\mathbf{v}$ (preservation of scale).

Examples of Vector Spaces over $\mathbb{R}$

To show that a set V is a vector space one has to state explicit rules defining

(i) scalar multiplication in V: $*_V : V \times \mathbb{R} \to V$

(ii) vector addition in V : $+_V : V \times V \to V$

(iii) the zero vector $\mathbf{0}_V$ in V: $\mathbf{0}_V + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$

and then verify that all 8 axioms are then satisfied by virtue of the explicit rules for (i), (ii), and (iii).

Example 1. $V = \mathbb{R}^n = \{[x_1, \ldots, x_n] \mid x_1, \ldots, x_n \in \mathbb{R}\}$

- scalar multiplication : $\lambda [x_1, \ldots, x_n] \equiv [\lambda x_1, \ldots, \lambda x_n]$
- vector addition :
  $[x_1, \ldots, x_n] + [y_1, \ldots, y_n] = [x_1 + y_1, \ldots, x_n + y_n]$
- zero vector : $\mathbf{0} = [0, \ldots, 0]$

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 2. Subspaces

Let V be a subspace of $\mathbf{R}^n$. Since V is closed under both scalar multiplication and vector addition

- $\lambda \in \mathbb{R}\,,\ \mathbf{v} \in V \quad \Longrightarrow \quad (\lambda \mathbf{v}) \in V,$
- $\mathbf{v}_1, \mathbf{v}_2 \in V \quad \Longrightarrow \quad \mathbf{v}_1 + \mathbf{v}_2 \in V$

one obtains rules for scalar multiplication and vector addition in V by restricting operations of scalar multiplication and vector addition in $\mathbb{R}^n$ to V:

$$\begin{aligned} *_V \ &: \ \mathbb{R} \times V \to V \quad \equiv \quad *|_{\mathbb{R}\times V} \\ +_V \ &: \ V \times V \to V \quad \equiv \quad +|_{V\times V} \end{aligned}$$

Furthermore, one can use $\mathbf{0}_{\mathbb{R}^n} \in V$ as the zero-vector $\mathbf{0}_V$ of V. With $*_V$, $+_V$, and $\mathbf{0}_V$ so defined, all 8 axiom are satisfied, and so any subspace of $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 3. $V = \{$ functions $f : \mathbb{R} \to \mathbb{R}\}$

- scalar multiplication : $(\lambda f)(x) = \lambda f(x)$
- vector addition : $(f + g)(x) = f(x) + g(x)$
- zero vector : $\mathbf{0} =$ the function f_0 defined by $f_0(x) = 0$ for all $x \in \mathbb{R}$.

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 4. $V = \{\text{vibrational modes of a stretched string}\}$

- scalar multiplication : changing the amplitude of vibrations by a factor $|\lambda|$ and also reversing the phase of a vibration if $\lambda < 0$.
- vector addition: superimposing vibrational modes (like harmonics)
- zero vector : $\mathbf{0}$ = the string at rest

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 5. $V = \{\text{polynomials of degree } n\}$

- scalar multiplication :
  $\lambda\left(a_n x^n + \cdots + a_1 x + a_0\right) = \lambda a_n x^n + \cdots + \lambda a_1 x + \lambda a_0$
- vector addition :
  $(a_n x^n + \cdots + a_0) + (b_n x^n + \cdots + b_0) = (a_n + b_n) x^n + \cdots + (a_0 + b_0)$
- zero vector : the zero polynomial (all coefficients $= 0$).

Working with General Vector Spaces

From now on we shall think of the vector space $\mathbb{R}^n$ as a special case of these more general vector spaces over $\mathbb{R}$.

Yet, $\mathbb{R}^n$ shall remain fundamental, since it will continue to provide the concrete computational platform for general linear algebra.

I will show you how computations for general vector spaces are carried out a little latter in this lecture.
First, however, I want to show how that the definitions and structural results we had the vector space $\mathbb{R}^n$ extend to these more general vector spaces **by simply replacing the vector space $\mathbf{R}^n$ with a general vector space V over $\mathbb{R}$.**

Revised Definitions

Definition

A **subspace** of a general vector space V is a subset $W \subset V$ that is closed under both the operations of scalar multiplication and vector addition:

$$\begin{aligned} \lambda \in \mathbf{R}, \mathbf{w} \in w \quad &\Longrightarrow \quad \lambda \mathbf{w} \in W \\ \mathbf{w}_1, \mathbf{w}_2 \in W \quad &\Longrightarrow \quad \mathbf{w}_1 + \mathbf{w}_2 \in W \end{aligned}$$

Definition

The **span** of a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in a general vector space V is the subspace of V generated by these vectors:

$$span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k) \equiv \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots c_k \in \mathbb{R}\}$$

Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in a general vector space V are **linearly independent** if

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}_V \quad \Longleftrightarrow \quad x_1 = 0, \ldots, x_k = 0$$

Definition

A **basis** for a subspace W of a general vector space V is a set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ such that

- $W = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$
- $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ are linearly independent

Theorem

- *Every vector space has a basis.*
- *Every basis for a vector space has the same number of vectors*

Definition

The **dimension** of a vector space V is the number of vectors in any basis for V.

Definition

A **linear transformation** between two vector spaces V and W is a function $T : V \to W$ such that

$$\begin{aligned} T(\lambda \mathbf{v}) &= \lambda T(\mathbf{v}) \\ T(\mathbf{v}_1 + \mathbf{v}_2) &= T(\mathbf{v}_1) + T(\mathbf{v}_2) \end{aligned}$$

Definition

The **range** of a linear transformation $T : V \to W$ is the subspace of W defined by

$$Range(T) \equiv \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

Definition

The **kernel** of a linear transformation $T : V \to W$ is the subspace of V defined by

$$Kernel(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_V\}$$

Vector Spaces Isomorphisms

Definition

A linear transformation $T : V \to W$ is an **isomorphism** if there exists a linear transformation $T^{-1} : W \to V$ such that

$$\begin{aligned} T^{-1}(T(\mathbf{v})) &= \mathbf{v} \quad , \quad \forall \mathbf{v} \in V; \\ T\left(T^{-1}(\mathbf{w})\right) &= \mathbf{w} \quad , \quad \forall \mathbf{w} \in W \end{aligned}$$

Theorem

A linear transformation $T : V \to W$ is an isomorphism **if and only if**

(i) *$Range(T) = W$, and*

(ii) *$Kernel(T) = \{\mathbf{0}_V\}$*

Finite Dimensional Vector Spaces

In this course, we restrict our attention to finite-dimensional vector spaces.
Here is a fundamental result concerning finite dimensional vector spaces:

Theorem

Every finite dimensional vector space V is isomorphic to some $\mathbb{R}^n$.

Proof. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V.

I'll now establish an invertible linear transformation $i_B : V \to \mathbb{R}^n$.

Proof, Cont'd

Because B is a basis for V, every vector $\mathbf{v} \in V$ has a unique expression as

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \qquad (*)$$

For any $\mathbf{v} \in V$, we define $i_B : V \to \mathbb{R}^n$ by

$$i_B\left(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n\right) \equiv [c_1, \ldots, c_n] \in \mathbb{R}^n$$

(the numbers $c_1, \ldots, c_n$ being uniquely determined by the expansion (*)).
Clearly, $\mathit{Range}\left(i_B\right) = \mathbb{R}^n$, since we can reach any vector $[x_1, \ldots, x_n] \in \mathbb{R}^n$ by applying i_B to the vector $\mathbf{v} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n \in V$

Proof, Cont'd

It is also clear that

$$i_B\left(c_1\mathbf{b}_1+\cdots+c_n\mathbf{b}_n\right)=[0,\ldots,0]\in\mathbb{R}^n \quad\Longrightarrow\quad c_1=0,\ldots,c_n=0$$

and so

$$Kernel\left(i_B\right)=\{\mathbf{0}_V\}$$

Since

(i) $Range\left(i_B\right)=\mathbb{R}^n$,

(ii) $Kernel\left(i_B\right)=\{\mathbf{0}_V\}$

Thus, $i_B : V\to\mathbb{R}^n$ is an isomorphism and so V and $\mathbb{R}^n$ are isomorphic.

$\square$.

Coordinatizing General Vector Spaces

Unlike $\mathbb{R}^n$ which comes equipped with a natural basis

$$\mathcal{E} = \{[1,0,0,\ldots,0,0],[0,1,0,\ldots,0,0],\ldots,[0,0,0,\ldots,0,1]\}$$

(the **standard basis** for $\mathbb{R}^n$), general vector spaces usually do not such an obvious basis.

Nevertheless, **bases are vital to doing calculations in a general vector space**.

However, once one has a basis $B = \{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ for a general vector space V, the isomorphism

$$i_B : V \to \mathbb{R}^n ; i_B\left(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n\right) \equiv [c_1,\ldots,c_n]$$

provides unique numerical coordinates for every vector $\mathbf{v} \in V$.

The Calculation Scheme for General Vector Spaces

Given a general vector space V with basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, one carries out calculations by using the isomorphism $i_B : V \to \mathbb{R}^n$ to translate questions about vectors, subspaces, etc. in V to questions about vectors, subspaces, etc. in $\mathbb{R}^n$ (our fundamental calculational platform).

Then when one knows the answers to the questions in $\mathbb{R}^n$, one can use the inverse linear transformation $i_B^{-1} : \mathbb{R}^n \to V$ to recover the corresponding answer in the context of vectors, subspaces, etc. in V.

$$
\begin{array}{ccc}
\text{computation in } V & \xrightarrow{\ i_B\ } & \text{computation in } \mathbb{R}^n \\
 & & \downarrow \\
\text{answer in } V & \xleftarrow{\ i_B^{-1}\ } & \text{answer in } \mathbb{R}^n
\end{array}
$$

Application: Solutions of Homogeneous Linear Differential Equations

We've seen that the set of functions on the real line can be given the structure of a vector space over $\mathbb{R}$.
I'll now show you how results from Math 2233 Differential Equations can be understood linear algebraically.
Let $C^{\infty}(\mathbb{R})$ be the set of differentiable functions on the real line.
$C^{\infty}(\mathbb{R})$ together with the operations

$$\begin{aligned}(\lambda f)(x) &= \lambda f(x)\\ (f+g)(x) &= f(x)+g(x)\end{aligned}$$

satisfies the axioms of a generalized vector space.

Application: $C^\infty(\mathbb{R})$, Cont'd

Consider the derivative operator $\frac{d}{dx}$. It sends functions in $C^\infty(\mathbb{R})$ to functions in $C^\infty(\mathbb{R})$. Moreover, it is compatible with the operations of scalar multiplication and vector addition:

$$\begin{aligned} \frac{d}{dx}(\lambda f) &= \lambda \frac{df}{dx} \qquad \text{if } \lambda \in \mathbb{R} \\ \frac{d}{dx}(f+g) &= \frac{df}{dx} + \frac{dg}{dx} \end{aligned}$$

Thus,

$$\frac{d}{dx} : C^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$$

is actually a linear transformation.
Similarly, all higher derivatives are also linear transformations from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$.

Now consider the differential equation

$$\frac{d^2 f}{dx^2} = 0$$

In linear algebraic language, finding the solutions of $\frac{d^2 f}{dx^2} = 0$ is equivalent to finding the kernel of the linear transformation $\frac{d^2}{dx^2}$

$$\ker\left(\frac{d^2}{dx^2}\right) = \left\{ f \in C^{\infty(\mathbb{R})} \mid \frac{d^2}{dx^2} f = 0 \right\}$$

In Math 2233 (or even Calculus 1), one finds that the general solution of this differential equation is

$$f(x) = c_1 + c_2 x \qquad , \qquad c_1, c_2 \in \mathbb{R}$$

Thus,

$$\frac{d^2 f}{dx^2} = 0 \quad \Rightarrow \quad f(x) = c_1 \cdot 1 + c_2 \cdot x$$

Linear algebraically, we can interpret this result as saying the subspace $\ker\left(\frac{d^2}{dx^2}\right)$ is generated by two functions $f_1(x) = 1$ and $f_2(x) = x$.

In fact, $f_1(x) = 1$ and $f_2(x) = x$, are linearly independent, and so provide a basis for $\ker\left(\frac{d^2}{dx^2}\right)$.

Indeed, the Wronkskian Condition

$$0 \neq W[f_1, f_2](x) = f_1(x)\frac{df_2}{dx}(x) - \left(\frac{df_1}{dx}(x)\right) f_2(x)$$

that one encounters in Math 2233, is just the differential equations method for checking that two functions are linearly independent.

For if the Wronskian condition does not hold

$$0 = W[f_1, f_2] = f_1(x)\frac{df_2}{dx}(x) - \left(\frac{df_1}{dx}(x)\right) f_2(x) = 0$$

we get a first order ordinary differential equation for $f_2(x)$ whose solution is

$$f_2(x) = \lambda f_1(x) \quad \text{for some } \lambda \in \mathbb{R}$$

and this in turn implies that

$$\lambda f_1(x) - f_2(x) = 0$$

and so the two solutions would not be linearly independent (and so would not provide a basis for the solution set).