Lecture 23: General Vector Spaces: Definitions and Examples

Math 3013
Oklahoma State University

March 23, 2022

Agenda

1. General Vector Spaces
2. More Examples and Applications

Vector Spaces over $\mathbb{R}$

Definition

A **vector space over** $\mathbb{R}$ is a set V for which the following operations are defined

- **scalar multiplication**: for every $\lambda \in \mathbb{R}$ and $\mathbf{v} \in V$ we have a map $(\lambda, \mathbf{v}) \to \lambda\mathbf{v} \in V$.
- **vector addition:** for every pair of vectors $\mathbf{u}, \mathbf{v} \in V$ we have a map $(\mathbf{u}, \mathbf{v}) \to \mathbf{u} + \mathbf{v} \in V$

It is also necessary that these operations of vector addition and scalar multiplication together satisfy 8 axioms.

Vector Space Axioms

1. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associativity of vector addition)
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (commutativity of vector addition)
3. There exists an element $\mathbf{0}\in V$ such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in V$. (additive identity.)
4. For each $\mathbf{v}\in V$ there exists an element $-\mathbf{v}\in V$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$. (additive inverses)
5. $\lambda(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\lambda\mathbf{v}$ (distributivity of scalar multipliciation over vector addition).
6. $(\lambda+\mu)\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}$. (distributivity of scalar multiplication over addition of scalars)
7. $\lambda(\mu\mathbf{v})=(\lambda\mu)\mathbf{v}$ (scalar multiplication preserves associativity of multiplication in $\mathbb{R}$.)
8. $(1)\mathbf{v}=\mathbf{v}$ (preservation of scale).

Examples of Vector Spaces over $\mathbb{R}$

To show that a set V is a vector space one has to state explicit rules defining

(i) scalar multiplication in V: $*_V : V \times \mathbb{R} \to V$

(ii) vector addition in V : $+_V : V \times V \to V$

(iii) the zero vector $\mathbf{0}_V$ in V: $\mathbf{0}_V + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$

and then verify that all 8 axioms are then satisfied by virtue of the explicit rules for (i), (ii), and (iii).

Example 1. $V = \mathbb{R}^n = \{[x_1, \dots, x_n] \mid x_1, \dots, x_n \in \mathbb{R}\}$

- scalar multiplication : $\lambda [x_1, \dots, x_n] \equiv [\lambda x_1, \dots, \lambda x_n]$
- vector addition :
  $[x_1, \dots, x_n] + [y_1, \dots, y_n] = [x_1 + y_1, \dots, x_n + y_n]$
- zero vector : $\mathbf{0} = [0, \dots, 0]$

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 2. Subspaces

Let V be a subspace of $\mathbf{R}^n$. Since V is closed under both scalar multiplication and vector addition

- $\lambda \in \mathbb{R}\,,\ \mathbf{v} \in V \quad \Longrightarrow \quad (\lambda \mathbf{v}) \in V,$
- $\mathbf{v}_1, \mathbf{v}_2 \in V \quad \Longrightarrow \quad \mathbf{v}_1 + \mathbf{v}_2 \in V$

one obtains rules for scalar multiplication and vector addition in V by restricting operations of scalar multiplication and vector addition in $\mathbb{R}^n$ to V:

$$\begin{aligned} *_V \ &: \ \mathbb{R} \times V \to V \quad \equiv \quad *|_{\mathbb{R}\times V} \\ +_V \ &: \ V \times V \to V \quad \equiv \quad +|_{V\times V} \end{aligned}$$

Furthermore, one can use $\mathbf{0}_{\mathbb{R}^n} \in V$ as the zero-vector $\mathbf{0}_V$ of V. With $*_V$, $+_V$, and $\mathbf{0}_V$ so defined, all 8 axiom are satisfied, and so any subspace of $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 3. $V = \{$ functions $f : \mathbb{R} \to \mathbb{R}\}$

- scalar multiplication : $(\lambda f)(x) = \lambda f(x)$
- vector addition : $(f + g)(x) = f(x) + g(x)$
- zero vector : $\mathbf{0} =$ the function f_0 defined by $f_0(x) = 0$ for all $x \in \mathbb{R}$.

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 4. $V = \{\text{vibrational modes of a stretched string}\}$

- scalar multiplication : changing the amplitude of vibrations by a factor $|\lambda|$ and also reversing the phase of a vibration if $\lambda < 0$.
- vector addition: superimposing vibrational modes (like harmonics)
- zero vector : $\mathbf{0}$ = the string at rest

Examples of Vector Spaces over $\mathbb{R}$, Cont'd

Example 5. $V = \{\text{polynomials of degree } n\}$

- scalar multiplication :
  $\lambda\left(a_n x^n + \cdots + a_1 x + a_0\right) = \lambda a_n x^n + \cdots + \lambda a_1 x + \lambda a_0$
- vector addition :
  $(a_n x^n + \cdots + a_0) + (b_n x^n + \cdots + b_0) = (a_n + b_n) x^n + \cdots + (a_0 + b_0)$
- zero vector : the zero polynomial (all coefficients $= 0$).

Working with General Vector Spaces

From now on we shall think of the vector space $\mathbb{R}^n$ as a special case of these more general vector spaces over $\mathbb{R}$.

Yet, $\mathbb{R}^n$ shall remain fundamental, since it will continue to provide the concrete computational platform for general linear algebra.

I will show you how computations for general vector spaces are carried out a little latter in this lecture.
First, however, I want to show how that the definitions and structural results we had the vector space $\mathbb{R}^n$ extend to these more general vector spaces **by simply replacing the vector space $\mathbf{R}^n$ with a general vector space V over $\mathbb{R}$.**

Revised Definitions

Definition

A **subspace** of a general vector space V is a subset $W \subset V$ that is closed under both the operations of scalar multiplication and vector addition:

$$\lambda \in \mathbb{R}, \mathbf{w} \in w \quad \Longrightarrow \quad \lambda \mathbf{w} \in W$$
$$\mathbf{w}_1, \mathbf{w}_2 \in W \quad \Longrightarrow \quad \mathbf{w}_1 + \mathbf{w}_2 \in W$$

Definition

The **span** of a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in a general vector space V is the subspace of V generated by these vectors:

$$span(\mathbf{v}_1, \ldots, \mathbf{v}_k) \equiv \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots c_k \in \mathbb{R}\}$$

Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in a general vector space V are **linearly independent** if

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}_V \quad \iff \quad x_1 = 0, \ldots, x_k = 0$$

Definition

A **basis** for a subspace W of a general vector space V is a set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ such that

- $W = span(\mathbf{w}_1, \ldots, \mathbf{w}_k)$
- $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ are linearly independent

Theorem

- *Every vector space has a basis.*
- *Every basis for a vector space has the same number of vectors*

Definition

The **dimension** of a vector space V is the number of vectors in any basis for V.

Examples of Bases in More General Vector Spaces

Recall that **bases** provides us with a means of coordinatizing a vector space.

N.B. Such a coordinatization is necessary for explicit computations.

Let's recall that procedure:

Suppose V is a vector space (e.g., a subspace of $\mathbb{R}^n$) and $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V. Then

• Every vector $\mathbf{v} \in V$ can be uniquely written as

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

• The naturally ordered list of numbers $c_1, \ldots, c_n$ can be used to form a *purely numerical vector*

$$\mathbf{v}_B = [c_1, \ldots, c_n] \in \mathbb{R}^n$$

• We can thus use such numerical vectors as "coordinate vectors" for the abstract vectors V.

Examples of Bases in More General Vector Spaces, Cont'd

Example 1: Taylor Expansions
In Calculus II, one learns that every smooth function on the real line has a unique Taylor expansion:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0) + \cdots$$

You can now understand this situation as saying

The vector space of smooth functions on the real line has as a basis

$$B = \left\{1, x - x_0, (x - x_0)^2, (x - x_0)^3, \ldots\right\}$$

Definition

A **linear transformation** between two vector spaces V and W is a function $T : V \to W$ such that

$$\begin{aligned} T(\lambda \mathbf{v}) &= \lambda T(\mathbf{v}) \\ T(\mathbf{v}_1 + \mathbf{v}_2) &= T(\mathbf{v}_1) + T(\mathbf{v}_2) \end{aligned}$$

Definition

The **range** of a linear transformation $T : V \to W$ is the subspace of W defined by

$$Range(T) \equiv \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

Definition

The **kernel** of a linear transformation $T : V \to W$ is the subspace of V defined by

$$Kernel(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$$

Example of an Important Linear Transformation on a Vector Space of Functions

Let $C^\infty(\mathbb{R})$ be the vector space of smooth functions on the real line.
Since

$$\begin{aligned} \lambda &\in \mathbb{R}\,,\ f \in C^\infty(\mathbb{R}) &\Rightarrow& \quad \frac{d}{dx}(\lambda f) = \lambda \frac{df}{dx} \\ f, g &\in C^\infty(\mathbb{R}) &\Rightarrow& \quad \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \end{aligned}$$

Thus, taking derivatives w.r.t. x defines a linear transformation

$$\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$

Vector Spaces Isomorphisms

Definition

A linear transformation $T : V \to W$ is an **isomorphism** if there exists a linear transformation $T^{-1} : W \to V$ such that

$$\begin{aligned} T^{-1}\left(T\left(\mathbf{v}\right)\right) &= \mathbf{v} \quad , \quad \forall \mathbf{v} \in V; \\ T\left(T^{-1}\left(\mathbf{w}\right)\right) &= \mathbf{w} \quad , \quad \forall \mathbf{w} \in W \end{aligned}$$

Theorem

A linear transformation $T : V \to W$ *is an isomorphism* **if and only if**

(i) *Range*$(T) = W$, *and*

(ii) *Kernel*$(T) = \{\mathbf{0}_V\}$

Finite Dimensional Vector Spaces

In this course, we restrict our attention to finite-dimensional vector spaces.
Here is a fundamental result concerning finite dimensional vector spaces:

Theorem

Every finite dimensional vector space V is isomorphic to some $\mathbb{R}^n$.

Proof. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V.

I'll now establish an invertible linear transformation $i_B : V \to \mathbb{R}^n$.

Proof, Cont'd

Because B is a basis for V, every vector $\mathbf{v} \in V$ has a unique expression as

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \qquad (*)$$

For any $\mathbf{v} \in V$, we define $i_B : V \to \mathbb{R}^n$ by

$$i_B\left(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n\right) \equiv [c_1, \ldots, c_n] \in \mathbb{R}^n$$

(the numbers $c_1, \ldots, c_n$ being uniquely determined by the expansion (*)).
Clearly, $Range\left(i_B\right) = \mathbb{R}^n$, since we can reach any vector $[x_1, \ldots, x_n] \in \mathbb{R}^n$ by applying i_B to the vector $\mathbf{v} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n \in V$

Proof, Cont'd

It is also clear that

$$i_B\left(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n\right) = [0, \ldots, 0] \in \mathbb{R}^n \quad \Longrightarrow \quad c_1 = 0, \ldots, c_n = 0$$

and so

$$Kernel\left(i_B\right) = \{\mathbf{0}_V\}$$

Since

(i) $Range\left(i_B\right) = \mathbb{R}^n$,

(ii) $Kernel\left(i_B\right) = \{\mathbf{0}_V\}$

Thus, $i_B : V \rightarrow \mathbb{R}^n$ is an isomorphism and so V and $\mathbb{R}^n$ are isomorphic.

$\square$.

Coordinatizing General Vector Spaces

Unlike $\mathbb{R}^n$ which comes equipped with a natural basis

$$\mathcal{E} = \{[1,0,0,\ldots,0,0],[0,1,0,\ldots,0,0],\ldots,[0,0,0,\ldots,0,1]\}$$

(the **standard basis** for $\mathbb{R}^n$), general vector spaces usually do not such an obvious basis.

Nevertheless, **bases are vital to doing calculations in a general vector space**.

However, once one has a basis $B = \{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ for a general vector space V, the isomorphism

$$i_B : V \to \mathbb{R}^n; i_B\left(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n\right) \equiv [c_1,\ldots,c_n]$$

provides unique numerical coordinates for every vector $\mathbf{v} \in V$.

The Calculation Scheme for General Vector Spaces

Given a general vector space V with basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, one carries out calculations by using the isomorphism $i_B : V \to \mathbb{R}^n$ to translate questions about vectors, subspaces, etc. in V to questions about vectors, subspaces, etc. in $\mathbb{R}^n$ (our fundamental calculational platform).

Then when one knows the answers to the questions in $\mathbb{R}^n$, one can use the inverse linear transformation $i_B^{-1} : \mathbb{R}^n \to V$ to recover the corresponding answer in the context of vectors, subspaces, etc. in V.

$$\begin{array}{ccc} \text{computation in } V & \xrightarrow{\;i_B\;} & \text{computation in } \mathbb{R}^n \\ & & \downarrow \\ \text{answer in } V & \xleftarrow{\;i_B^{-1}\;} & \text{answer in } \mathbb{R}^n \end{array}$$

Application: Solutions of Homogeneous Linear Differential Equations

We've seen that the set of functions on the real line can be given the structure of a vector space over $\mathbb{R}$.
I'll now show you how results from Math 2233 Differential Equations can be understood linear algebraically.
Let $C^{\infty}(\mathbb{R})$ be the set of differentiable functions on the real line.
$C^{\infty}(\mathbb{R})$ together with the operations

$$\begin{aligned} (\lambda f)(x) &= \lambda f(x) \\ (f+g)(x) &= f(x)+g(x) \end{aligned}$$

satisfies the axioms of a generalized vector space.

Application: $C^\infty(\mathbb{R})$, Cont'd

Recall that the derivative operator $\frac{d}{dx}$. is compatible with the operations of scalar multiplication and vector addition in $C^\infty(\mathbb{R})$:

$$\begin{aligned} \frac{d}{dx}(\lambda f) &= \lambda \frac{df}{dx} \qquad \text{if } \lambda \in \mathbb{R} \\ \frac{d}{dx}(f+g) &= \frac{df}{dx} + \frac{dg}{dx} \end{aligned}$$

Thus,

$$\frac{d}{dx} : C^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$$

defines a linear transformation $\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

Similarly, all higher derivatives are also linear transformations from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$.

Now consider the differential equation

$$\frac{d^2 f}{dx^2} = 0$$

In linear algebraic language, finding the solutions of $\frac{d^2 f}{dx^2} = 0$ is equivalent to finding the kernel of the linear transformation $\frac{d^2}{dx^2}$

$$\ker\left(\frac{d^2}{dx^2}\right) = \left\{ f \in C^{\infty(\mathbb{R})} \mid \frac{d^2}{dx^2} f = 0 \right\}$$

In Math 2233 (or even Calculus 1), one finds that the general solution of this differential equation is

$$f(x) = c_1 + c_2 x \qquad , \qquad c_1, c_2 \in \mathbb{R}$$

Thus,

$$\frac{d^2 f}{dx^2} = 0 \quad \Rightarrow \quad f(x) = c_1 \cdot 1 + c_2 \cdot x$$

Linear algebraically, we can interpret this result as saying the subspace $\ker\left(\frac{d^2}{dx^2}\right)$ is generated by two functions $f_1(x) = 1$ and $f_2(x) = x$.

In fact, $f_1(x) = 1$ and $f_2(x) = x$, are linearly independent, and so provide a basis for $\ker\left(\frac{d^2}{dx^2}\right)$.
Indeed, the Wronkskian Condition

$$0 \neq W[f_1, f_2](x) = f_1(x)\frac{df_2}{dx}(x) - \left(\frac{df_1}{dx}(x)\right) f_2(x)$$

that one encounters in Math 2233, is just the differential equations method for checking that two functions are linearly independent.
For if the Wronskian condition does not hold

$$0 = W[f_1, f_2] = f_1(x)\frac{df_2}{dx}(x) - \left(\frac{df_1}{dx}(x)\right) f_2(x) = 0$$

we get a first order ordinary differential equation for $f_2(x)$ whose solution is

$$f_2(x) = \lambda f_1(x) \quad \text{for some } \lambda \in \mathbb{R}$$

and this in turn implies that

$$\lambda f_1(x) - f_2(x) = 0$$

and so the two solutions would not be linearly independent (and so would not provide a basis for the solution set).